Facility Location Games

Yury Kochetov

Sobolev Institute of Mathematics. Novosibirsk. Russia

Contents

The *p*-median problem

- Input: J is the set of clients;

  I is the set of potential facilities;

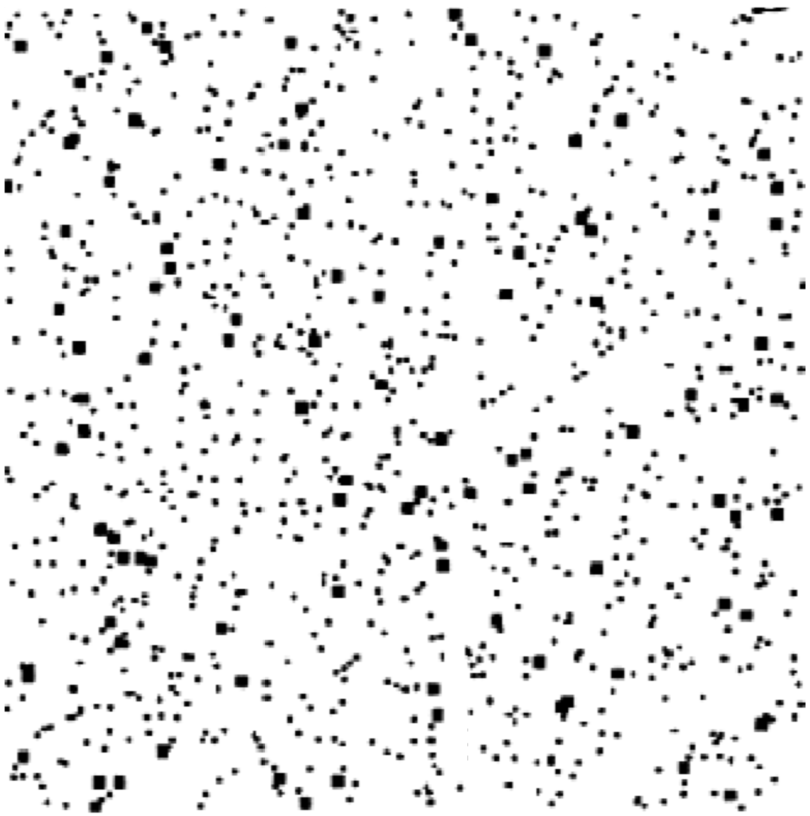
  c_{ij} is the distance for servicing client j from facility i;

  p is the number of opening facilities;

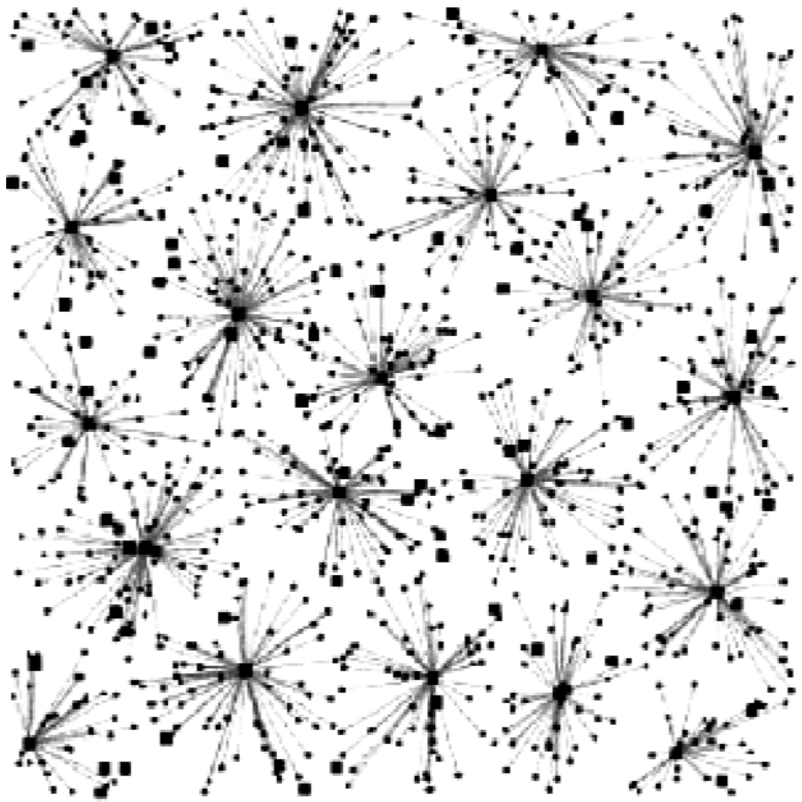
- Goal: to find a set $S \subset I$, $|S| = p$ of opened facilities in such way to minimize the total distance from the facilities to clients:

$$\min_{|S|=p} \{ \sum_{j \in J} \min_{i \in S} c_{ij} \}$$

Example $|I| = 100; |J| = 1000$



Instance



Solution

The Leader-Follower Location Problem

- Input:
  - J is the set of clients;
  - I is the set of potential facilities;
  - w_j is the demand of client j;
  - c_{ij} is the distance from client j to facility i;
  - p is the number of leader facilities;
  - r is the number of follower facilities.

Each client patronizes the closest opened facility.

- Output: a set $S \subset I$, $|S| = p$ of opening facilities by the leader.

- Goal: maximize the market share of the leader anticipating that the follower will react to the decision by opening his own r facilities.

Decision variables

$$x_i = \begin{cases} 1 & \text{if the leader opens facility } i \\ 0 & \text{otherwise} \end{cases}$$

$$y_i = \begin{cases} 1 & \text{if the follower opens facility } i \\ 0 & \text{otherwise} \end{cases}$$

$$z_j = \begin{cases} 1 & \text{if client } j \text{ patronizes a leader facility} \\ 0 & \text{if client } j \text{ patronizes a follower facility} \end{cases}$$

For given solution x we introduce the set

$$I_j(x) = \{i \in I | c_{ij} < \min_{l \in I} c_{lj} | x_l = 1\}$$

of facilities which allow the follower to “capture” client j.

The Bi-Level 0-1 Linear Program

$$\max_{x} \sum_{j \in J} w_j z_j^*(x)$$

s.t.

$$\sum_{i \in I} x_i = p, \qquad x_i \in \{0,1\}, i \in I,$$

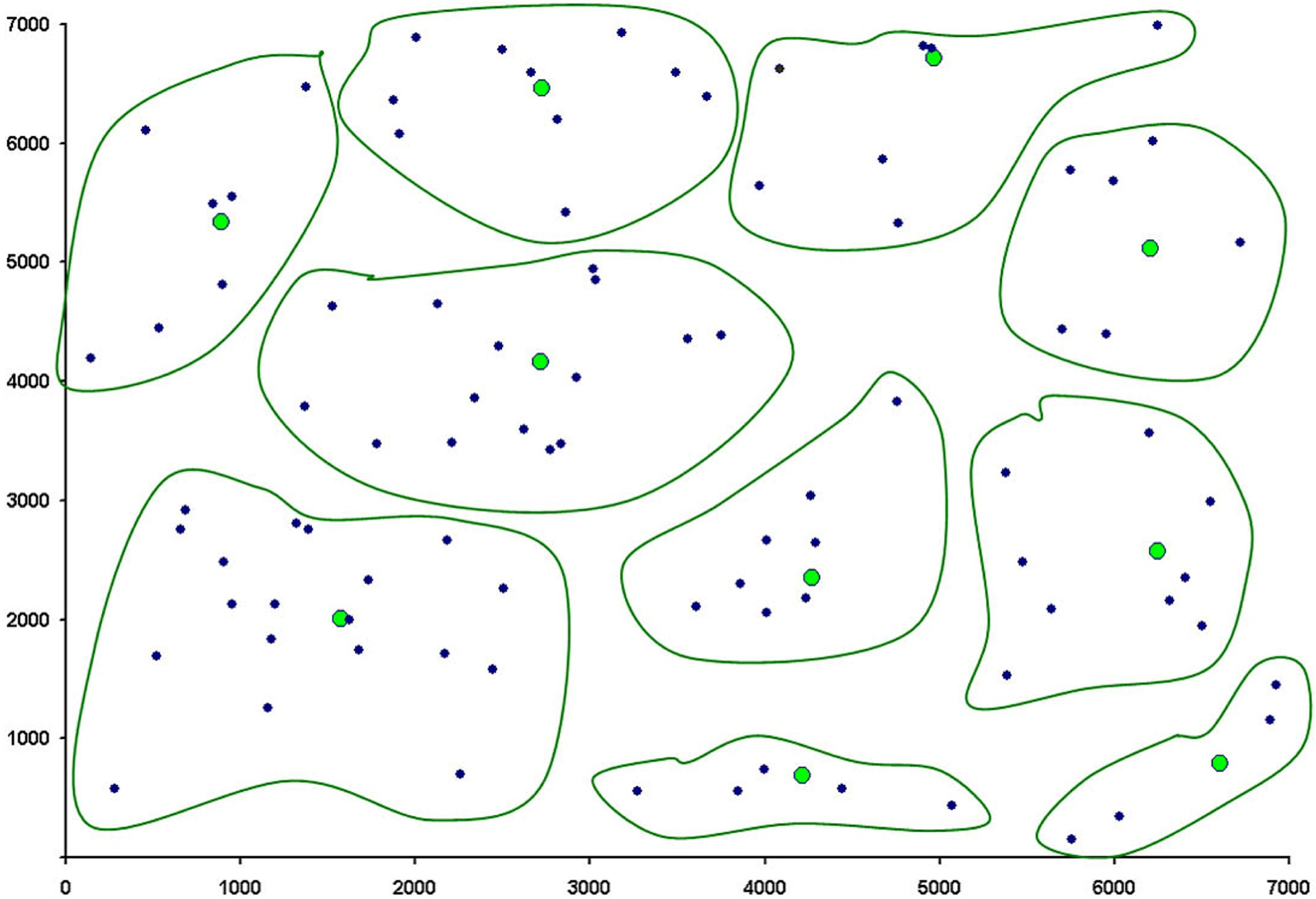
where z_j^* is optimal solution of the Follower problem

$$\max_{z,y} \sum_{j \in J} w_j (1 - z_j)$$

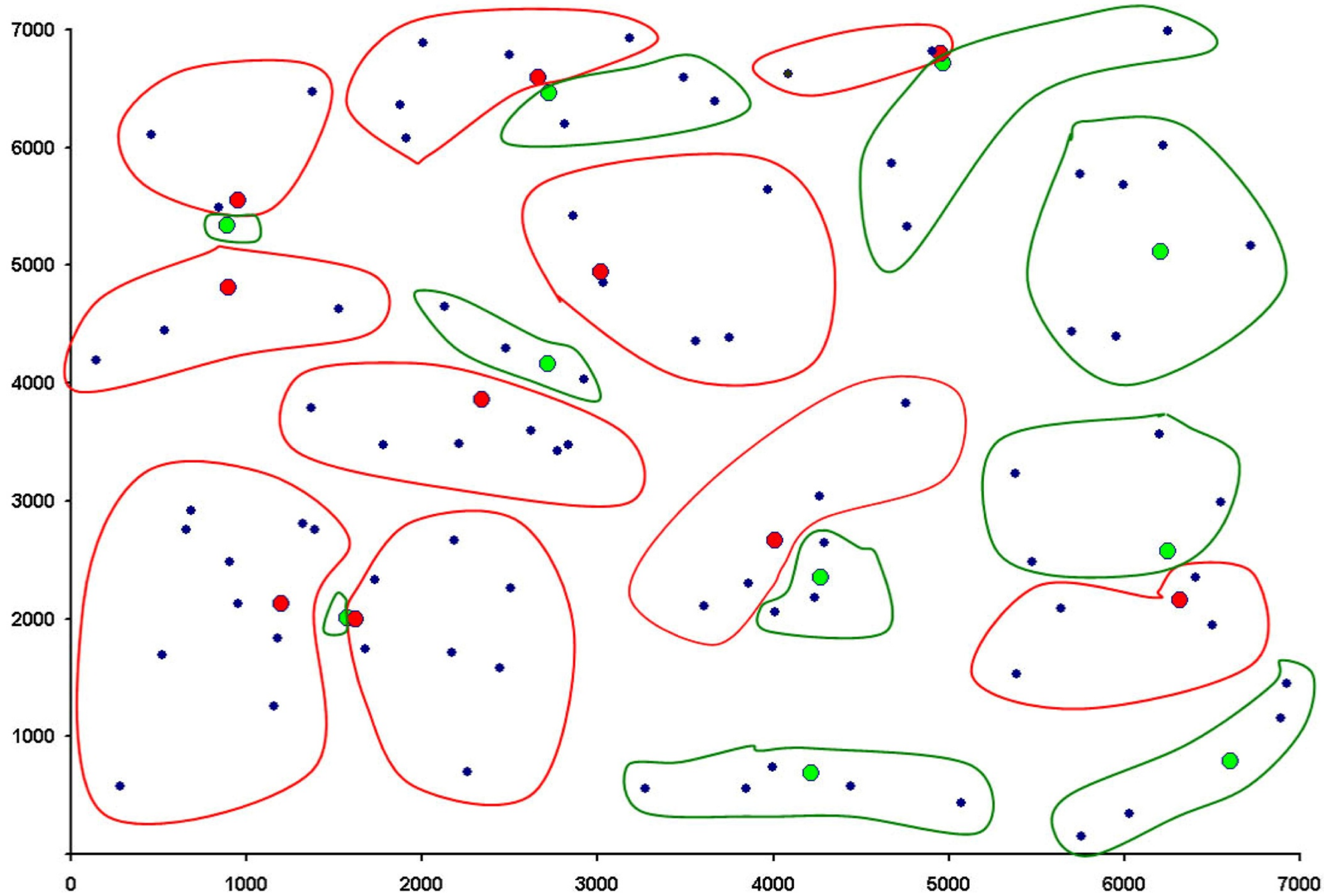
$$1 - z_j \le \sum_{i \in I_j(x)} y_i\,, \qquad j \in J;$$

$$\sum_{i \in I} y_i = r, \qquad x_i + y_i \le 1, i \in I, \qquad y_i, z_j \in \{0,1\}.$$

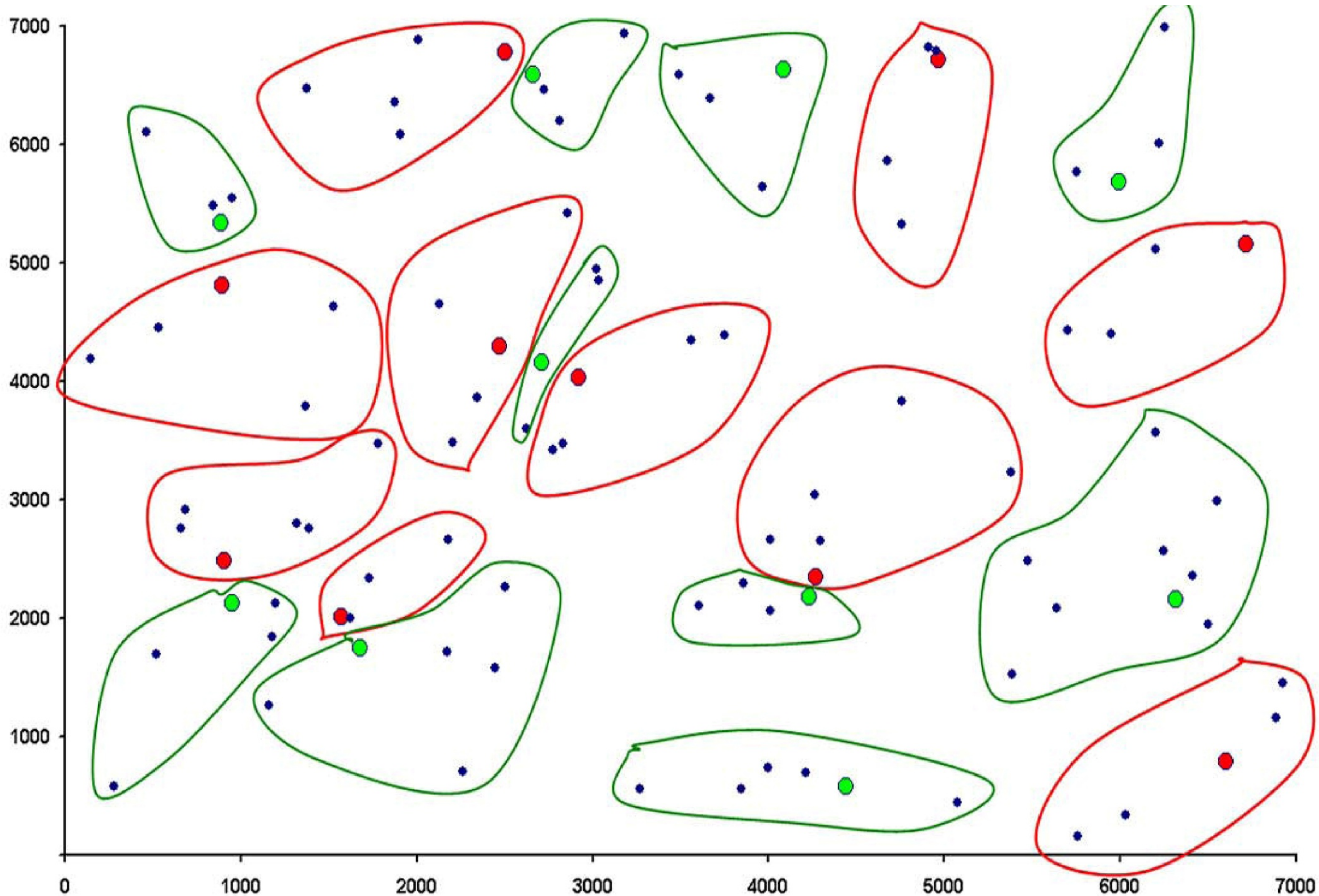
The leader ignores the follower



Optimal solution of the follower. Market share of the leader is 41 %



Optimal solution of the leader. Market share of the leader is 50 %



Theoretical and Empirical Results

- $\sum_2^P$ −hard problem even for Euclidean distances (I. Davydov, E. Carrizosa, Yu. Kochetov, 2012)
- The follower problem is NP–hard in the strong sense (I. Davydov, E. Carrizosa, Yu. Kochetov, 2012)
- Pollynomially solvable cases (J. Spoerhase, H.C. Wirth, H. Noltemeir, 2007)
- The branch and cut method (M.C. Roboredo, A.A. Pessoa, 2012)
- An iterative exact method (E. Alekseeva, Yu. Kochetov, A. Plyasunov)
- Metaheuristics (E. Alekseeva et al. 2010; D. Serra, C. ReVelle, 1995; I. Davydov, 2012; J.A. Moreno Perez et al., 2009)

Exact method

Decision variables

$$x_i = \begin{cases} 1 & \text{if the leader opens facility } i \\ 0 & \text{otherwise} \end{cases}$$

$$z_{ij} = \begin{cases} 1 & \text{if the leader facility } i \text{ is closest to client } j \\ 0 & \text{otherwise} \end{cases}$$

D is the market share of the leader

Notations:

$\mathcal{F}$ is nonempty family of follower solutions.

For $y \in \mathcal{F}$ we define the set $I_j(y)$ of the facilities which allow to the leader saving client j:

$$I_j(y) = \{i \in I | c_{ij} \le \min_{l \in I} c_{lj} | y_l = 1\}$$

The Single Level Reformulation

$$\max D$$

$$\text{s.t.} \quad \sum_{j \in J} \sum_{i \in I_j(y)} w_j z_{ij} \geq D, \quad y \in \mathcal{F}$$

$$\sum_{i \in I} z_{ij} = 1, \quad j \in J$$

$$x_i \geq z_{ij}, \qquad i \in I, j \in J$$

$$\sum_{i \in I} x_i = p$$

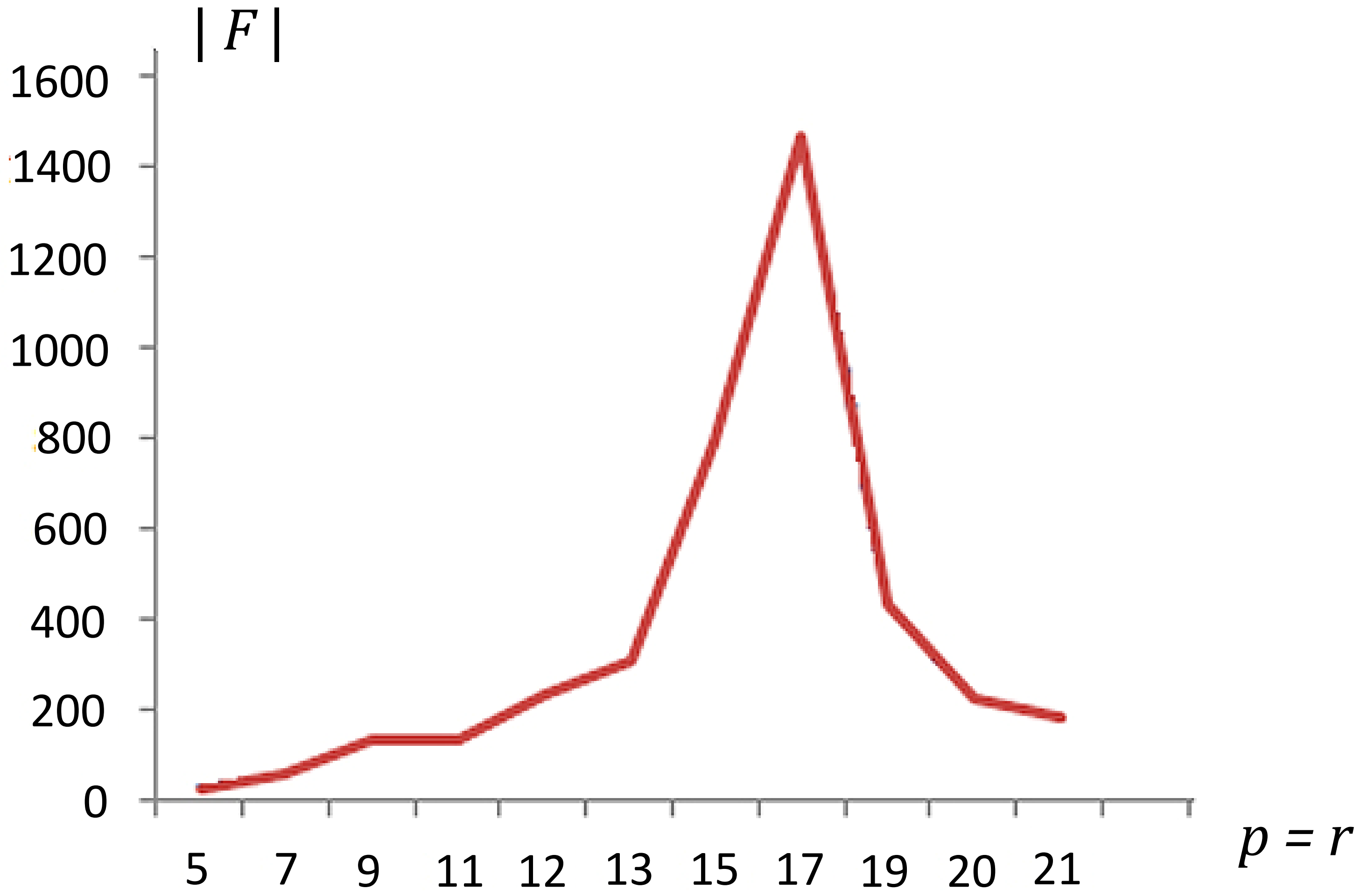
$$x_i, z_{ij} \in \{0,1\}, \quad D \geq 0$$

If $\mathcal{F}$ contains all follower solutions, we have an equivalent reformulation.

Iterative Exact Method

0. Choose an initial subfamily $F \in \mathcal{F}$ and put $D^* := 0$.
1. Solve the problem with F instead of $\mathcal{F}$ and find $x(F)$ and upper bound $D(F)$.
2. Solve the follower problem for $x(F)$ and find $y(F)$ and lower bound $D(y)$.
3. If $D^* < D(y)$ then $D^* := D(y)$.
4. If $D^* = D(F)$ then STOP.
5. Include $y(F)$ into the subfamily F and go to 1.

The total number of iterations depending on the parameters *p* and *r*, *n* = *m* = 50, class *Euclidean*



The Leader-Follower Facility Location and Design Problem

Leader enters in a market by opening own facilities.

Follower already has own facilities and reacts by opening new facilities, closing existing ones, and adjusting the attractiveness of its existing facilities.

Each client patronizes a facility proportionally to the attractiveness of the facility and inversely proportionally to the distance between client and the facility (Huff's gravity-based rule).

The objective of each firm is to find out the optimal location and attractiveness of the facilities in such a way that its own profit is maximized.

Parameters

$J = \{1, \dots, n\}$ is the set of clients;

$I = \{1, \dots, m\}$ is the set of candidate facilities of the leader;

$K = \{1, \dots, r_1\}$ is the set of existing facilities of the follower;

$L = \{1, \dots, r_2\}$ is the set of candidate facilities of the follower;

Parameters

w_j	buying power of client j
c_i	unit attractiveness cost of leader's facility i
e_l	unit attractiveness cost of follower's facility l
b_k	unit cost of changing attractiveness of follower's facility k
f_i	fixed cost of opening facility i by the leader
$\tilde{f}_l$	fixed cost of opening facility l by the follower
t_k	revenue of closing an existing facility k
U_i	maximal attractiveness of leader's facility i
$\overline{M}_l$	maximal attractiveness of follower's facility l
$\overline{A}_k$	maximal attractiveness of existing follower's facility k
$\underline{A}_k$	current attractiveness of existing follower's facility k

Decision Variables

$$x_i = \begin{cases} 1 & \text{if facility } i \text{ is opened by the leader} \\ 0 & \text{otherwise} \end{cases}$$

Q_i is attractiveness of facility i of the leader;

$$z_k = \begin{cases} 1 & \text{if existing facility } k \text{ is kept open by the follower} \\ 0 & \text{otherwise} \end{cases}$$

A_k new attractiveness of existing facility k;

$$y_l = \begin{cases} 1 & \text{if new facility } l \text{ is opened by the follower} \\ 0 & \text{otherwise} \end{cases}$$

M_l is attractiveness of new facility l of the follower.

The gravity based rule

Q_i/d_{ij}^2 is the utility of facility i with attractiveness Q_i for client j;

$\sum_{k \in K} \frac{A_k}{\bar{d}_{kj}^2} + \sum_{l \in L} \frac{M_l}{\tilde{d}_{lj}^2}$ is the total utility of the follower facilities for client $\boldsymbol{j}$;

The probability that client j visit a facility i is expressed as

$$p_{ij} = \frac{Q_i / d_{ij}^2}{\sum_{i \in I} Q_i / d_{ij}^2 + \sum_{k \in K} A_k / \bar{d}_{kj}^2 + \sum_{l \in L} M_l / \tilde{d}_{lj}^2}$$

Bi-Level Model

$$\max_{x,Q} \sum_{j\in J} w_j \sum_{i\in I} p_{ij} - \sum_{i\in I} f_i x_i - \sum_{i\in I} c_i Q_i$$

s.t.

$$Q_i \le U_i x_i, \quad i \in I;$$

$$Q_i > 0, \ \ x_i \in \{0, 1\}, \quad i \in I;$$

$$\max_{z,y,A,M} \sum_{j\in J} w_j (1 - \sum_{i\in I} p_{ij}) + \sum_{k\in K} t_k (1 - z_k) - \sum_{k\in K} b_k (A_k - \underline{A}_k z_k) - \sum_{l\in L} e_l M_l - \sum_{l\in L} \tilde{f}_l y_l$$

s.t.

$$A_k \le \bar{A}_k z_k, \quad k \in K;$$

$$M_l \le \bar{M}_l y_l, \quad l \in L;$$

$$A_k \ge 0, \ \ M_l \ge 0, \ \ z_k, y_l, \in \{0, 1\}, \quad k \in K, l \in L.$$

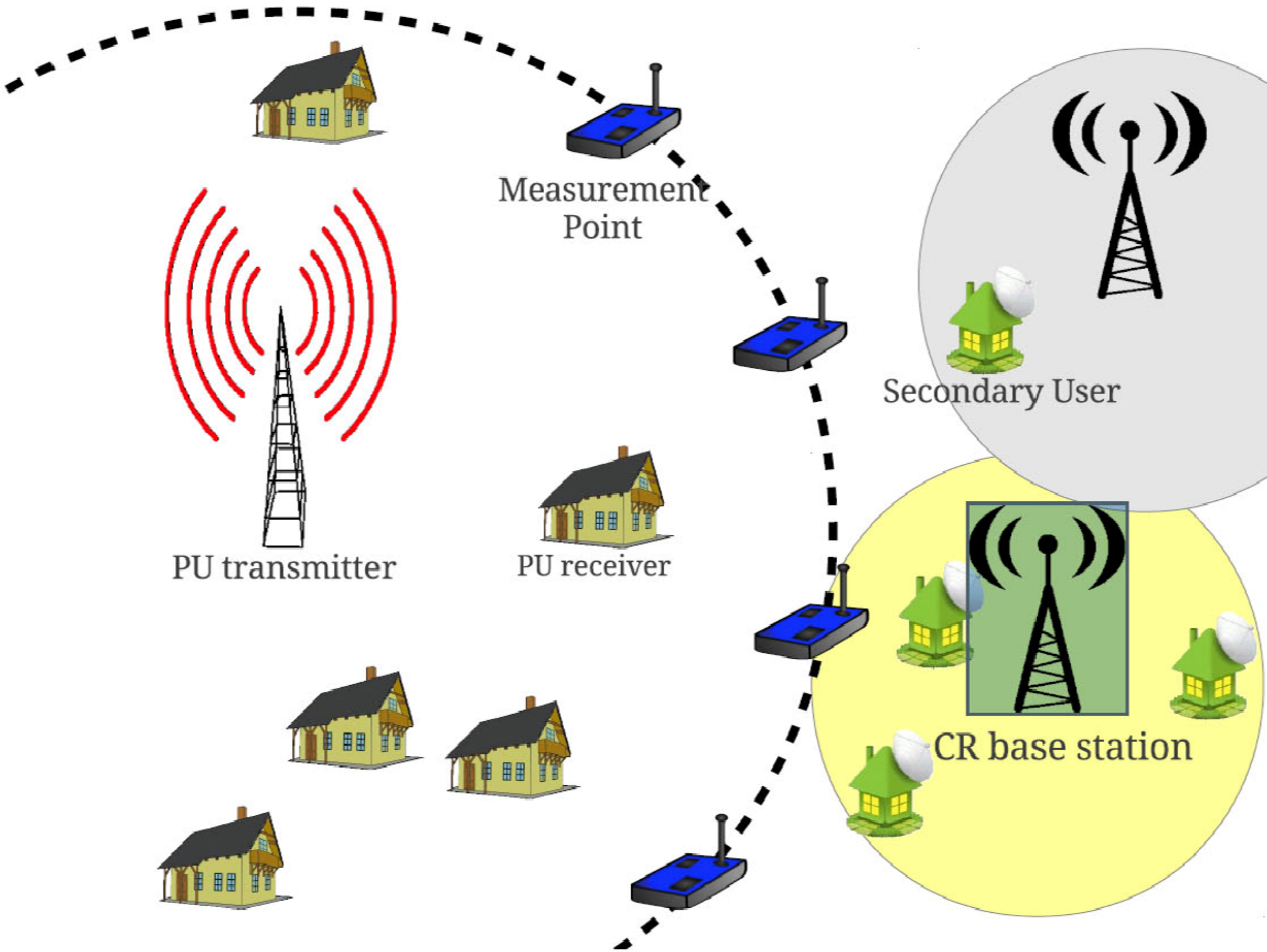
Strategic Planning in Cognitive Radio Networks

We consider a primary network operating on a set of frequency bands.

A cognitive radio operator (the leader) wants to deploy a durable secondary network by opportunistically using the unused capacity of the primary network. To this end, the operator places a set of own base stations and tunes the correspondent transmission power so as to maximize the profit drawn from the served clients.

The operator has to:

- ensure that the deployment of the secondary network does not impair the primary network;
- pay for each base station under the budget constraint;
- find a solution which will be robust face to the arrival of a possible competitor (the follower).


Measurement
Point
Secondary User
PU transmitter
PU receiver
CR base station

Leader Problem

$$LP: \max \left\{ \sum_{j\in N} w_j \sum_{i\in B_s} \sum_{c\in C} x_{ijc} - \lambda \sum_{i\in B_s} \sum_{c\in C} x_{ic} \right\}$$

$$s.t. \quad \sum_{i\in B_s} \sum_{c\in C} (x_{ijc} + y^*_{ijc}) \leq 1 \qquad j \in J$$

$$\sum_{i\in B_s} \sum_{c\in C} x_{ic} \leq K$$

$$\sum_{c\in C} x_{ic} \leq 1 \qquad i \in B_s$$

$$P_{ic} \leq P_{max} x_{ic} \qquad (i,c) \in B_s \times C$$

$$x_{ijc} \leq x_{ic} \qquad (i,j,c) \in B_s \times J \times C$$

$$\sum_{i \in B_s} P_{ic} h_{imc} \leq \bar{I}_{mc} \qquad (m, c) \in B_m \times C$$

$$P_{ic} g_{ijc}^{i} \geq \ \bar{\gamma} \cdot \sum_{l \in B_s, l \neq i} P_{lc} g_{ljc}^{i} + \bar{\gamma} \cdot \sum_{r \in B_s} Q_{rc}^{*} g_{rjc}^{i} +$$

$$+ \bar{\gamma} \cdot \sum_{k \in B_p} H_{kc} l_{kjc}^{i} + \bar{\gamma} N_c - \Gamma\left(1 - x_{ijc}\right) \quad (i, j, c) \in B_s \times J \times C$$

$$x_{ic}, x_{ijc} \in \{0,1\}, \ \ P_{ic} \geq 0$$

Follower Problem

$$FP: \max \left\{ \sum_{j \in N} w_j \sum_{i \in B_s} \sum_{c \in C} y_{ijc} - \lambda \sum_{i \in B_s} \sum_{c \in C} y_{ic} \right\}$$

$$s.t. \quad \sum_{c \in C} (x_{ijc} + y_{ijc}) \leq 1 \qquad i \in B_s$$

$$\sum_{c \in C} y_{ic} \leq 1 \qquad i \in B_s$$

$$y_{ijc} \leq y_{ic} \qquad (i, j, c) \in B_s \times J \times C$$

$$Q_{ic} \leq Q_{max} y_{ic} \qquad (i, c) \in B_s \times C$$

$$\sum_{i \in B_s} Q_{ic} h_{imc} \leq \bar{I}_{mc} - I_{mc} \qquad (m, c) \in B_m \times C$$

$$Q_{ic} g^{i}_{ijc} \geq \bar{\gamma} \cdot \sum_{r \in B_s, r \neq i} Q_{rc} g^{i}_{rjc} + \bar{\gamma} \cdot \sum_{l \in B_s} P_{lc} g^{i}_{ljc} +$$

$$+ \bar{\gamma} \cdot \sum_{k \in B_p} H_{kc} l^{i}_{kjc} + \bar{\gamma} N_c - \Gamma(1 - y_{ijc}) \quad (i, j, c) \in B_s \times J \times C$$

$$Q_{ic} g_{ijc} \geq Q_{rd} g_{rjd} - \Gamma(1 - y_{ijc}) \quad (i, r) \in B_s, j \in J, (c, d) \in C$$

$$Q_{ic} g_{ijc} \geq P_{ld} g_{ljd} - \Gamma(1 - y_{ijc}) \quad (i, l) \in B_s \quad j \in J \quad (c, d) \in C$$

$$y_{ic}, y_{ijc} \in \{0,1\}, \quad Q_{ic} \geq 0$$

Theorem 1. The leader problem is Σ_2^P-hard.

Theorem 2. The follower problem is NP-hard in the strong sense.

We design a hybrid stochastic tabu search algorithm for this Stackelberg game. At each step, we solve the mixed integer program derived from the follower problem by CPLEX software.

Conclusions

- Bi-level facility location models are presented
- Recent results are reviewed
- New interesting models can be obtained:
  - using detail models for user behavior;
  - continuous locations;
  - prices and others.