ON FINDING CRITICAL INDEPENDENT AND VERTEX SETS

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Abstract. An independent set I_c of a undirected graph G is called critical if

$$|I_c| - |N(I_c)| = \max\{|I| - |N(I)| : I \text{ is an independent set of } G\},$$

where $N(I)$ is the set of all vertices of G adjacent to some vertex of I. It has been proved by Cun-Quan Zhang [*SIAM J. Discrete Math.*, 3 (1990), pp. 431–438] that the problem of finding a critical independent set is polynomially solvable. This paper shows that the problem can be solved in $O(|V(G)|^{1/2}|E(G)|)$ time and its weighted version in $O(|V(G)|^2|E(G)|^{1/2})$ time.

Key words. independent set, minimum cut

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1. Introduction. Denote by G a simple undirected graph and by $N(U)$, $U \subseteq V(G)$, the set of all vertices of G adjacent to some vertex of U. An independent set $I_c \subseteq V(G)$ is called *critical* if

$$\nu(G) = |I_c| - |N(I_c)| = \max\{|I| - |N(I)| : I \text{ is an independent set of } G\}.$$

The number $\nu(G)$ is a parameter of G, closely related to some other important ones [Zh90]. A vertex set U_c is called *critical* if

$$\mu(G) = |U_c| - |N(U_c)| = \max\{|U| - |N(U)| : U \subseteq V(G)\}.$$

Cun-Quan Zhang [Zh90] observed that $\nu(G) = \mu(G)$ and that the problem of finding critical independent set is reducible to the problem of finding critical vertex set. In [Zh90] it is also shown by rather a sophisticated reduction to the linear program that the problem of finding critical vertex set is polynomially solvable. This paper presents a very simple reduction of the weighted version of the problem to the maximum weight independent set problem on bipartite graphs. As a consequence, we obtain a fast algorithm to find critical independent and vertex sets in general graphs.

2. Critical weighted sets. Let G be a undirected graph with nonnegative weights $w(v)$ on its vertices. For any $X \subseteq V$, denote $\rho(X) = w(X) - w(N(X))$, where $w(S)$ is the total weight $\sum_{v \in S} w(v)$ of $S \subseteq V$.

A vertex set $U^* \subseteq V(G)$ is *critical weighted* if $\rho(U^*) = \max\{\rho(U) : U \subseteq V(G)\}$. An independent set I^* is *critical weighted* if

$$\rho(I^*) = \max\{\rho(I) : I \text{ is an independent set of } G\}.$$

Obviously, $\rho(I^*) \le \rho(U^*)$. We claim that $\rho(I^*) = \rho(U^*)$. Indeed, let U be a critical weighted vertex set. Define $I \subseteq U$ to be the set of all isolated vertices of the subgraph induced by U. Then, clearly, I is an independent set of G. Since $N(U) = U_1 \cup X$, where $U_1 = U \setminus I$, $X = (N(U_1) \setminus U_1) \cup N(I)$, we have

$$\rho(U) = w(U_1) + w(I) - w(U_1) - w(X) \le \rho(I).$$

It follows that $\rho(U) = \rho(I)$ and I is a critical weighted independent set of G.

Thus, to find a critical weighted independent set, it suffices to find a critical weighted vertex set.

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3. The reduction. Suppose that we have an instance of the problem of finding critical weighted vertex set. Let (x_v), (y_v) be $(0,1)$ vectors whose components are indexed by the vertices of G. Consider the following $(0,1)$ programming problem:

$$\max \sum_{u\in V} w(u)x_u - \sum_{v\in V} w(v)y_v, \tag{3.1}$$

$$\text{s. t.}\quad y_v \geq x_u,\quad (u,v)\in E(G), \tag{3.2}$$

$$x_u, y_v \in \{0,1\},\quad u,v\in V(G). \tag{3.3}$$

We claim that if (x_v^*), (y_v^*) is an optimal solution to this problem then (x_v^*) is the incidence vector of a critical weighted vertex set. Indeed, let $X^* = \{v\in V : x_v^* = 1\}$. Observe that (y_v^*) is the incidence vector of $N(X^*)$. Now let (x_v) be the incidence vector of some $X\subseteq V$ and (y_v) be the incidence vector of $N(X)$. Since $(x_v),(y_v)$ is a feasible solution of (3.1)–(3.3), it follows that

$$\rho(X) = \sum_{u\in V} w(u)x_u - \sum_{v\in V} w(v)y_v \leq \sum_{u\in V} w(u)x_u^* - \sum_{v\in V} w(v)y_v^* = \rho(X^*).$$

It is easy to see that (3.1)–(3.3) is an instance of the selection problem [Ba70], [Rh70]. Finding an optimal solution of (3.1)–(3.3) is known to be equivalent to finding a minimum cut in a bipartite network [Ba70], [Rh70], [PQ82]. Hence an optimal solution of (3.1)–(3.3) can be found in $O(|V(G)|^2|E(G)|^{1/2})$ time [GT88], [CM89]. Furthermore, putting $y_v = 1 - z_v$, $v\in V$, one can rewrite (3.1)–(3.3) as follows:

$$\max \sum_{u\in V} w(u)x_u + \sum_{v\in V} w(v)z_v + |V(G)|,$$

$$\text{s. t.}\quad x_u + z_v \leq 1,\quad (u,v)\in E(G),$$

$$x_u, z_v \in \{0,1\},\quad u,v\in V(G).$$

The latter is nothing but an instance of the maximum weight independent set problem on bipartite graph H with bipartition (V,V') where V' is a copy of V and for any $u\in V$, $v'\in V'$,

$$(u,v')\in E(H)\quad\text{if and only if}\quad (u,v)\in E(G).$$

In the case of unit weights ($w(v)\equiv 1$) an optimal solution to this can be obtained as a by-product with the $O(|V(G)|^{1/2}|E(G)|)$ algorithm for finding maximum matching in bipartite graphs [PS82, p.226].

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