On Finding the Maximum Number of Disjoint Cuts in Seymour Graphs*

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Abstract. In the CUT PACKING problem, given an undirected connected graph G, it is required to find the maximum number of pairwise edge disjoint cuts in G. It is an open question if CUT PACKING is NP-hard on general graphs. In this paper we prove that the problem is polynomially solvable on Seymour graphs which include both all bipartite and all series-parallel graphs. We also consider the weighted version of the problem in which each edge of the graph G has a nonnegative weight and the weight of a cut D is equal to the maximum weight of edges in D. We show that the weighted version is NP-hard even on cubic planar graphs.

1 Introduction

In the CUT PACKING problem, given an undirected connected graph G, it is required to find the maximum number of pairwise edge disjoint cuts in G. This problem looks natural and has various connections with many well-known optimization problems on graphs. E. g., one can observe that INDEPENDENT SET can be treated as a constrained version of CUT PACKING whose collection of feasible cuts consists of all stars. The directed counterpart of CUT PACKING — DICUT PACKING — is well known to be polynomially solvable on general graphs (see Lucchesi [14], Lucchesi and Younger [15], Frank [8] or Grötschel et al. [11, p. 252]). By contrast, the complexity status of CUT PACKING on general graphs still remains an open problem. In this paper we use a characterization in [1] and an algorithmic result on joins due to Frank [9] to prove that CUT PACKING is polynomially solvable when restricted to the family of Seymour graphs. To present a rigorous definition of Seymour graph we need to introduce a few more notions. At this point we only notice that the family includes both all bipartite and all series-parallel graphs (Seymour [18][19]). It should be also noted that the complexity status of the recognition problem for Seymour graphs remains unclear; it is known only that the problem belongs to co-NP [1]. All we have to say for ourselves is that our result also holds for recognizable in

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polynomial time subfamilies of Seymour graphs such as, e. g., above mentioned bipartite[1] and series-parallel graphs.

Graphs in the paper are undirected with possible multiple edges and loops. Given a graph G and $v \in V(G)$, $N_G(v)$ will denote the set of vertices adjacent to v in G. Any inclusion-wise minimal cut can be represented in the form $\delta_G(X)$, i. e., as the set of edges leaving a set $X \subseteq V(G)$. If the context allows we shall omit subscripts and write $N(v)$, $\delta(X)$.

A *join* in a graph G is a set of edges $J \subseteq E(G)$ such that each circuit of G has at most half of edges in J. Matchings and shortest paths between any two vertices are simple examples of joins. The term "join" originates from the classical notion of T-join introduced by Edmonds and Johnson in their seminal paper [7]. Given a graph G and a vertex subset $T \subseteq V(G)$ of even cardinality, a subset J of edges of G is called a *T-join* if the set of vertices having odd cardinality in the subgraph of G spanned by J coincides with T. It can be shown that any minimum cardinality T-join is a join and, moreover, any join is a minimum cardinality T-join where T is the set of vertices having odd cardinality in the subgraph of G spanned by J (Guan's lemma [12]). This establishes a one-to-one correspondence between joins and minimum cardinality T-joins of a graph.

Let $\{D_1, \dots, D_k\}$ be a collection of disjoint cuts in G. Pick an edge e_i in each cut D_i and set $J = \{e_1, \dots, e_k\}$. Since each cut and each circuit in a graph may have in common only an even number of edges , J is a join. If a join J in G can be represented in this way, we call $\{D_1, \dots, D_k\}$ a *complete packing* of J and say that J *admits a complete packing* .

There are graphs in which not every join admits a complete packing. A simple example is K_4. Any two cuts of K_4 intersect and therefore, every join admitting a complete packing in K_4 consists of at most one edge whereas any perfect matching of K_4 constitutes a join of cardinality 2.

A graph G is called a *Seymour graph* if every join of G admits a complete packing.

A co-NP characterization of Seymour graphs is provided by the following theorem which is an easy corollary of a stronger result in [1] (a bit shorter alternative proof can be found in [2]).

Let G be a graph and J be a join of G. We say that a circuit C of G is *J-saturated* if exactly half of the edges of C lie in J. A subgraph H of G is called *J-critical* if it is non-bipartite and constitutes the union of two J-saturated circuits. E.g., if $G = K_4$ and J is a perfect matching, then G itself is J-critical.

Theorem 1 (Ageev, Kostochka & Szigeti [1]). *A graph G is not a Seymour graph if and only if G has a join J and a J-critical subgraph having maximum degree* 3.

The "if part" of this theorem is easy and holds even when no bound on the maximum degree is assumed (see [1]).

[1] András Frank pointed out to me that the polynomial-time solvability in the case of bipartite graphs was shown earlier and in a different way by D. H. Younger [20].

To prove that CUT PACKING is polynomially solvable on Seymour graphs we shall use the above characterization and the following algorithmic result.

Theorem 2 (Frank [9]). *Given an undirected graph G, a join of maximum cardinality in G can be found in polynomial time.*

The running time of Frank's algorithm is bounded by $O(mn)$, where $n = |V(G)|$, $m = |E(G)|$ [9]. Let G be a Seymour graph and J be a join of G with maximum cardinality returned by Frank's algorithm. By the definition of Seymour graph J admits a complete packing $\{D_1, \dots, D_k\}$ where $k = |J|$. Since the size of any collection of disjoint cuts in G does not exceed k, the collection $\{D_1, \dots, D_k\}$ provides an optimal solution for CUT PACKING on G. The question remains how to find such a collection in polynomial time.

We shall prove that even a bit more can be done.

Theorem 3. *Given a Seymour graph G and a join J in G, a complete packing of J can be found in polynomial time.*

This contrasts with the well-known result of Middendorf and Pfeiffer [17] that, given a planar cubic graph G and a join J of G, it is NP-complete to determine whether J admits a complete packing.

We notice that alternative and a bit more sophisticated proofs of Theorem 3 can be extracted directly from the arguments in [1] and [2].

Our proof of Theorem 3 relies upon three basic facts. The first fact — Lemma 1 — states that the family of Seymour graphs is closed under the operation of star contraction of a graph. To prove this we use Theorem 1. Lemma 1 is of apparent interest irrelatively the scope of this paper. In [3] we develop the related subject to elaborate an alternative co-NP characterization of Seymour graphs. The second fact — Lemma 2 — is well known among "T-joins" experts and reveals a connection between the same operation and joins admitting complete packings. Unfortunately, no published proof of this important statement appeared. To make the paper self-contained we include such a proof in Section 3. The third fact is the classical result due to Edmonds and Johnson [7] that, given a graph G and an even vertex subset T, there exists a polynomial-time algorithm to find a T-join of G with minimum cardinality. We shall use the straightforward corollary of it and Guan's lemma: given a graph G and an edge subset J, one can decide in polynomial time if J is a join of G.

Together with the original setting it seems natural to test for NP-hardness a bit more general, weighted version of the problem (WEIGHTED CUT PACKING): given a graph G with nonnegative edge weights $w(e)$, find a collection of disjoint cuts $\{D_1, \dots, D_k\}$ in G maximizing

$$\sum_{i=1}^{k} \max\{w(e) : e \in D_i\}.$$

CUT PACKING is equivalent to that special case of WEIGHTED CUT PACKING in which all weights $w(e)$ are equal to 1. We shall demonstrate in the last section of this paper that WEIGHTED CUT PACKING is NP-hard even on cubic planar graphs.

2 Seymour graphs and star contractions

Since every bipartite graph is a Seymour graph and every other graph can be obtained by contraction of a bipartite graph, the family of Seymour graphs is not closed under the operation of contracting an edge. The operation that truly preserves the Seymour property is star contraction.

Let G be an undirected graph. We say that a graph H is obtained from G by *contracting a star* if H is a result of contracting edges of some star in G.

If X is a vertex subset of a graph G then G/X will denote a graph which is obtained from G by identifying (shrinking) all vertices in X (after this shrinking new multiple edges and loops may appear). Notice that no edge is deleted from G and so G/X has the same set of edges as G.

Lemma 1. *Every graph obtained from a Seymour graph by contracting a star is a Seymour graph.*

Proof. Suppose to the contrary that a Seymour graph G has a vertex u such that contracting the star at u results in a non-Seymour graph H. We can equivalently think of H as the subgraph $G/X - u$ of G/X where $X = N_G(u)$ (for an illustration see Fig. 1). Let ξ denote the vertex of G/X that is the result of shrinking the set X in G. Notice that by Theorem 1, G does not have any J'-critical subgraph for any join J' of G. Since H is not a Seymour graph, by Theorem 1 it has a join J and a J-critical subgraph $S = C_1 \cup C_2$ of maximum degree 3, where C_1 and C_2 are J-saturated circuits. Note that J is also a join of G. Note first that ξ lies in S; otherwise S would be a J-critical subgraph of G, which by Theorem 1 would contradict the assumption that G is a Seymour graph. Let S', C_1', and C_2' denote the subgraphs of G spanned by the edges of G lying in S, C_1, and C_2 respectively. Let $\{x_1, x_2, \ldots, x_k\}$ denote the subset of vertices of S' lying in X. Recall that S has maximum degree 3 and hence, by the definition of J-critical subgraph, each vertex of S has degree either 2 or 3. It follows that $1 \le k \le 3$. Note first that in fact $2 \le k \le 3$, since $k = 1$ means

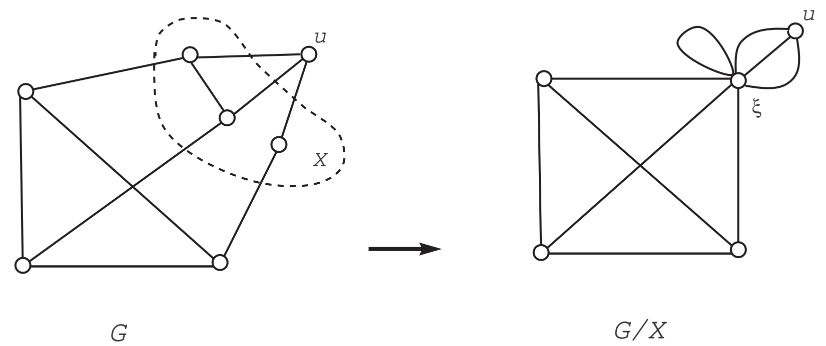

Fig. 1. A graph G and the graph G/X obtained from G by shrinking the vertex subset $X = N_G(u)$ to a new vertex ξ.

that S' is isomorphic to S which implies that S' is a J-critical subgraph of G, i. e. G is not a Seymour graph. Then at least one of the graphs C'_1 and C'_2 is a path. Recall that x_i are neighbours of u in G. For each i, let e_i denote an edge connecting x_i and u. Denote by T the subgraph of G obtained from S' by adding the vertex u and the edges e_i. Our goal in the remaining part of the proof is to show that T is a J^*-critical subgraph of G for some join J^*. By the remark just after Theorem 1 this contradicts the assumption that G is a Seymour graph and thus proves the lemma. Observe first that S can be obtained from T by contracting the star at u. Thus, since S is non-bipartite, T is non-bipartite as well.

Case 1: $k = 2$.
Let $J^* = J \cup \{e_1\}$. Of the circuits C_1 and C_2 at least one passes through ξ. Assume first that, say, C_1 does, whereas C_2 not. Then C'_1 is a path, whereas C'_2 is a circuit in G. Set $C''_1 = C'_1 \cup \{e_1\} \cup \{e_2\}$. Then, by construction, $T = C''_1 \cup C'_2$ and, moreover, C''_1 and C'_2 are both J^*-saturated circuits. Assume now that C'_1 and C'_2 are both paths. Then set $C''_i = C'_i \cup \{e_1\} \cup \{e_2\}$, $i = 1, 2$. Again, by construction, $T = C''_1 \cup C''_2$, and moreover, C''_1 and C''_2 are both J^*-saturated circuits. Thus, in either case T is a J^*-critical subgraph of G.

Case 2: $k = 3$ (for an illustration refer to Fig. 1).
In this case ξ has degree 3 in S' and, consequently, C_1 and C_2 both pass through ξ. Therefore C'_1 and C'_2 are both paths in G. Since $S = C_1 \cup C_2$, it follows that of the three edges incident with ξ in S, exactly one lies in both C_1 and C_2. W.l.o.g. we may assume that this edge is incident with x_1 in G. Our assumption implies that x_1 is a common endpoint of C'_1 and C'_2 and we may assume further that the other endpoint of C'_1 is x_2 whereas that of C'_2 is x_3. Now set $J^* = J \cup \{e_1\}$, $C''_1 = C'_1 \cup \{e_1\} \cup \{e_2\}$, and $C''_2 = C'_2 \cup \{e_1\} \cup \{e_3\}$. Then, by construction, $T = C''_1 \cup C''_2$ and moreover, C''_1 and C''_2 are both J^*-saturated circuits. It means that, again, T is a J^*-critical subgraph of G. □

3 Joins admitting complete packings and star contractions

In this section to make the paper self-contained we give a proof for a folklore lemma which is one of the crucial points in our proof of Theorem 3.

Let G be a graph. For $v \in V(G)$, denote by $G \star v$ the graph that is obtained from G by contracting the star at v. Let J be a join of G. A vertex $v \in V(G)$ is called *J-marginal* if v is incident with exactly one edge e in J and $J \setminus \{e\}$ is a join of $G \star v$.

Lemma 2. *Let G be a graph and J be a join of G. If J admits a complete packing, then G has a J-marginal vertex.*

Proof. Since J admits a complete packing, G has a collection of edge disjoint cuts such that each cut contains exactly one edge in J. Among all such collections choose a collection $\mathcal{C} = \{\delta(X_1), \delta(X_2), \ldots, \delta(X_k)\}$ with minimum $|X_1|$. For each

i, let $J \cap X_i = \{e_i\}$. Let v be the end of e_1 that lies in X_1. We claim that $X_1 = \{v\}$. Assume to the contrary that $X_1 \setminus \{v\} \neq \emptyset$. We show first that for any i, $X_1 \cap X_i$ is either $\emptyset$ or X_1. Assume not, that is $0 < |X_1 \cap X_i| < |X_1|$ for some i. Since $e_1 \notin \delta(X_i)$, either both ends of e_1 lie in X_i or both not; for the similar reason the same holds with respect to e_i and X_1. Note that we may assume that both ends of e_1 lie in X_i, since the replacement of the set X_i by its complement does not alter the collection $\mathcal{C}$. Assume that both ends of e_i do not lie in X_1. Set $X_1' = X_1 \cap X_i$ and $X_i' = X_1 \cup X_i$. It is easy to check that $\delta(X_1') \cap \delta(X_i') = \emptyset$, $\delta(X_1') \cup \delta(X_i') = \delta(X_1) \cup \delta(X_i)$, and $e_1 \in \delta(X_1')$, $e_i \in \delta(X_i')$. Replace in the collection $\mathcal{C}$ the cuts $\delta(X_1)$ and $\delta(X_i)$ by $\delta(X_1')$ and $\delta(X_i')$ respectively. This yields a new collection $\mathcal{C}' = \{\delta(X_1'), \delta(X_2), \ldots, \delta(X_i'), \ldots, \delta(X_k)\}$ with the same properties except that $|X_1'| < |X_1|$, which contradicts the choice of $\mathcal{C}$. Assume now that both ends of e_i lie in X_1. Then we set $X_1' = X_1 \cap \overline{X}_i$ and $X_i' = X_1 \cup \overline{X}_i$. Similarly, $\delta(X_1') \cap \delta(X_i') = \emptyset$, $\delta(X_1') \cup \delta(X_i') = \delta(X_1) \cup \delta(X_i)$, and, unlike the above, $e_1 \in \delta(X_i')$, $e_i \in \delta(X_1')$. This implies that, again, the collection $\mathcal{C}$ can be replaced by another collection having the same properties except that it includes the cut $\delta(X_1')$ with $|X_1'| < |X_1|$, which is a contradiction to the choice of $\mathcal{C}$. Thus we have proved that for each $i \neq 1$, either $X_1 \cap X_i = \emptyset$ or $X_1 \subset X_i$. It immediately follows that $X_1 = \{v\}$, since otherwise we could replace in $\mathcal{C}$ the cut $\delta(X_1)$ by the star cut $\delta(v)$. Set $S = N_G(v) \cup \{v\}$. Now consider the graph $G \star v$ and denote by ω the vertex of $G \star v$ that is the result of contracting the star at v in G. Since the cuts in $\mathcal{C}$ are edge disjoint, we have either $S \cap X_i = \emptyset$ or $S \subset X_i$ for each $i \neq 1$. Now for each $i \neq 1$, set $X_i^* = (X_i \setminus S) \cup \{\omega\}$ if $S \subset X_i$ and $X_i^* = X_i$ otherwise. It follows that $\mathcal{C}^* = \{\delta(X_2^*), \delta(X_3^*), \ldots, \delta(X_k^*)\}$ is a collection of disjoint cuts of $G \star v$ with $e_i \in \delta(X_i^*)$ for each $i \neq 1$. This implies that $J \setminus \{e_1\}$ is a join in $G \star v$ and $\mathcal{C}^*$ is a complete packing of $J \setminus \{e_1\}$, as desired. □

4 Proof of Theorem 3

We describe a polynomial-time algorithm that, given a Seymour graph G and a join J of G, finds a complete packing of J.

Let G be a graph and J be a join of G. If v is a J-marginal vertex of G, then $e(v)$ will denote the (unique) edge in J that is incident with v. The algorithm runs through k similar steps where $k = |J|$, and outputs a complete packing $\{D_1, D_2, \ldots, D_k\}$.

Step 0. Set $G_1 := G$, $J_1 := J$.

Step i, $1 \leq i \leq k$. Look successively through the vertices of G_i to find a J_i-marginal vertex v_i of G_i. To check if a vertex v of G_i is J_i-marginal, call one of the known polynomial-time algorithms that can test if $J_i \setminus \{e(v)\}$ is a join of $G_i \star v$. Set D_i to be equal to the set of edges of the star at v_i in G_i. Set $G_{i+1} := G_i \star v_i$, $J_{i+1} := J_i \setminus \{e(v_i)\}$. If $i < k$ go to step $i+1$, otherwise end up.

We now establish the correctness of the algorithm. By Lemma 1 each graph G_i is a Seymour graph, since each was obtained from the Seymour graph G by a sequence of star contractions. Using this, Lemma 2 and a simple induction on i we further obtain that, for each $i = 1, \ldots, k$, J_i is a join admitting a complete packing in G_i. Moreover, by Lemma 2, for each $i = 1, \dots, k$, G_i does contain a J_i-marginal vertex and thus the algorithm never terminates before step k. Next, it is clear from the description that D_i and D_j are disjoint whenever $i \neq j$. And finally, since every cut of G_i is a cut of G, each D_i is a cut of G. □

It is clear that the running time of the algorithm is $O(n^2 C(m,n))$ where $n = |V(G)|$, $m = |E(G)|$ and $C(m,n)$ is the running time of the chosen procedure to test if a given set of edges is a join. As such, one can use any of the known algorithms for finding a minimum cardinality T-join: the original $O(n^3)$-algorithm of Edmonds and Johnson ([7]), the $O(mn \log n)$-algorithm of Barahona et al. ([4],[6]) or an $O(n^{3/2} \log n)$-algorithm for the special case of planar Seymour graphs ([16],[10],[5]).

As was already mentioned above, the running time of Frank's algorithm for finding a join of maximum cardinality is $O(mn)$ [9]. Thus, if we use, e. g., the algorithm of Barahona et al. ([4],[6]) for finding a minimum cardinality T-join, the overall running time of our algorithm for solving CUT PACKING on Seymour graphs will be bounded by $O(mn^3 \log n)$.

5 NP-hardness result

Theorem 4. *WEIGHTED CUT PACKING is NP-hard on the cubic planar graphs.*

Proof. We use a reduction from the following NP-complete decision problem [17]: given a graph G and a join J of G, to decide if a join J admits a complete packing in G. Assign $(0,1)$-weights w to the edges of G in the following way: $w(e) = 1$ if $e \in J$ and $w(e) = 0$ otherwise. Let $\{D_1, D_2, \dots, D_k\}$ be a collection of edge disjoint cuts in G of maximum total w-weight. We may assume that each D_i in the collection has weight 1 and thus the total weight of the collection is k. We claim then that $k = |J|$ if and only if J admits a complete packing. Indeed, if $k = |J|$ then $\{D_1, D_2, \dots, D_k\}$ is a complete packing of J. On the other hand, if J admits a complete packing $\{D'_1, D'_2, \dots, D'_{|J|}\}$ then this packing has weight $|J|$. However, any collection of disjoint cuts in G has weight at most $|J|$ and so $k = |J|$. □

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References

1. Ageev, A., Kostochka, A., Szigeti, Z.: A characterization of Seymour graphs. J. Graph Theory **24** (1997) 357–364.
2. Ageev, A.A., Kostochka, A.V.: Vertex set partitions preserving conservativeness. Preprint 97-029, Bielefeld University, 1997. Submitted for journal publication.

3. Ageev, A., Sebő, A.: An alternative co-NP characterization of Seymour graphs. In preparation.
4. Barahona, F.: Applicationde l'optimisationcombinatuare a certainesmodeles de verres de spins: complexite et simulation. Master's thesis, Universite de Grenoble, France, 1980.
5. Barahona, F.: Planar multicommodity flows, max cut and the Chinese postman problem. In: Polyhedral Combinatorics, DIMACS Series in Discrete Mathematics and Computer Science **1** (1990) 189–202.
6. Barahona, F., Maynard, R., Rammal, R., Uhry, J.P.: Morphology of ground states of a 2-dimensional frustration model. Journal of Physics A: Mathematical and General **15** (1982) 673–699.
7. Edmonds, J., Johnson, E.L.: Matchings, Euler tours and the Chinese postman problem, Math. Programming **5** (1973) 88–124.
8. Frank, A: How to make a digraph strongly connected. Combinatorica **1** (1981) 145–153.
9. Frank, A: Conservative weightings and ear-decompositions of graphs. Combinatorica **13** (1993) 65–81.
10. Gabow, H.N.: A scaling algorithm for weighting matching in general graphs. In: 26th Annual Symposium on Foundations of Computer Science, IEEE, N.Y., 1985, 90–100.
11. Grötschel, M., Lovász, L., Schrijver, A.: Geometric algorithms and combinatorial optimization, Springer-Verlag, 1988.
12. Guan, M.G.: Graphic programming using odd and even points. Chinese Mathematics **1** (1962) 273–277.
13. Lovász, L., Plummer, M.D.: Matching Theory. Akadémiai Kiadó, Budapest, 1986.
14. Lucchesi, C.L.: A minimax equality for directed graphs, Ph.D. Thesis, University of Waterloo, Waterloo, Ontario, 1976.
15. Lucchesi, C.L., Younger, D.H.: A minimax theorem for directed graphs. J. of London Math. Soc. (2) **17** (1978) 369–374.
16. Matsumoto, K., Nishizeki, T., Saito, N.: Planar multicommodity flows, maximum matchings and negative cycles. SIAM J. on Computing **15** (1986) 495–510.
17. Middendorf, M., Pfeiffer, F.: On the complexity of the disjoint paths problem. Combinatorica **13** (1993) 97–107.
18. Seymour, P.D.: The matroids with the max-flow min-cut property. J. Comb. Theory B**23** (1977) 189–222.
19. Seymour, P.D.: On odd cuts and plane multicommodity flows. Proc. London Math. Soc. Ser. (3) **42** (1981) 178–192.
20. Younger, D.H.: Oral communication (1993).