Every Circle Graph of Girth at Least 5 is 3-Colourable

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Abstract

It is known that every triangle-free (equivalently, of girth at least 4) circle graph is 5-colourable (Kostochka, 1988) and that there exist examples of these graphs which are not 4-colourable (Ageev, 1996). In this note we show that every circle graph of girth at least 5 is 2-degenerate and, consequently, not only 3-colourable but even 3-choosable.

Keywords: circle graph, chromatic number, girth

1 Introduction

The *girth* of a graph G is the length of a shortest circuit of G. A graph G is a *circle graph* if it is isomorhic to the intersection graph of chords of a circle. Let Γ_k $(k = 3, 4, \dots)$ denote the family of circle graphs with girth at least k and let $\chi(\Gamma_k) = \{\max \chi(G) : G \in \Gamma_k)$ where $\chi(G)$ stands for the chromatic number of G. Since Γ_3 contains all complete graphs, $\chi(\Gamma_3) = \infty$. The problem of evaluating $\chi(\Gamma_4)$ (the maximum of chromatic number over all triangle-free circle graphs) has been independently posed in [2] and [4] (see also [3], p. 158). The ultimate result is that $\chi(\Gamma_4) = 5$; the upper bound is due to A. Kostochka [5], the lower bound to the author [1]. In this note we prove that $\chi(\Gamma_k) = 3$ for all remaining $k \geq 5$. In fact a much stronger statement holds.

A graph G is called *k-degenerate* $(k = 1, 2, \dots)$ if each induced subgraph of G has a vertex of degree at most k.

Theorem. *Every circle graph of girth at least 5 is 2-degenerate.*

A graph G with lists $L(v)$, associated with its vertices $v \in V(G)$, is called *L-list colourable* if there is a proper vertex colouring of G in which the colour assigned to a vertex v is chosen from $L(v)$. A graph G is *k-choosable* if there is at least one L-list colouring for every possible list assignment L with $|L(v)| = k$ for all $v \in V(G)$. By definition k-choosability implies k-colourability. A standard inductive argument shows that every k-degenerate graph is $(k+1)$-choosable. Therefore we have

Corollary. *Every circle graph of girth at least 5 is 3-choosable.* □

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2 Proof

Let I_1 and I_2 be two intervals of the real line. We say that I_1 *overlaps* I_2 or I_1 and I_2 *overlap* if $I_1 \cap I_2 \neq \emptyset$ and neither $I_1 \subseteq I_2$ nor $I_2 \subseteq I_1$. By definition a graph G is a circle graph if and only if one can associate with each vertex v of G an interval $I(v) = (h(v), t(v))$ of the real line in such a way that $uv \in E(G)$ if and only if $I(u)$ and $I(v)$ overlap. We call such a representation of a circle graph an *interval representation.* W.l.o.g. we may assume that no two intervals of an interval representation coincide.

Let G be a circle graph and $\mathcal{F} = \{I(v) : v \in V(G)\}$ be a fixed interval representation of G. Let H be a connected induced subgraph of G with at least two vertices. Then the union of all intervals in $\mathcal{F}$ corresponding to the vertices of H is an interval (a, b). Let $x_1 = a < x_2 < \ldots < x_{k-1} < x_k = b$ be the endpoints of those intervals. Define the *base* $B(H)$ of H to be the interval (x_2, x_{k-1}). Let $\beta(H)$ denote the length of the interval $B(H)$.

The first lemma is obvious.

Lemma 1. *Let G be a connected circle graph with an interval representation $\{I(v) : v \in V(G)\}$. Let I be an interval overlapping the interval $\bigcup_{v \in V(G)} I(v)$. Then I overlaps $I(v)$ for some v.* □

Lemma 2. *Let $P = v_1 v_2 \ldots v_k$ be a path. Let $\{I(v) : v \in V(P)\}$ be an interval representation of P such that $I(v_1) \not\subset I(v_j)$ for all $j \neq 1$ and $I(v_k) \not\subset I(v_l)$ for all $l \neq k$. Then $I(v_j) \not\subset I(v_l)$ for all $j \neq l$. Moreover, if $h(v_1) < h(v_k)$, then $h(v_1) < h(v_2) < t(v_1) < h(v_3) < t(v_2) < h(v_4) < \ldots < t(v_k)$ (see Figure 1).*

Proof. The latter statement obviously follows from the former, which we prove by induction on the length k of the path. Notice first that for $k \leq 3$ the statement is trivial. Consider the path $v_2 \ldots v_k$. If $I(v_2)$ were contained in another interval then either so would do $I(v_1)$, or the degree of v_1 would be at least 2, contradicting the assumptions of the lemma. Hence, by the induction hypothesis, $I(v_j) \not\subset I(v_l)$ for all $j, l \in \{2, \ldots, k\}$, $j \neq l$. Finally, if $I(v_1)$ contained another interval then again, by Lemma 1, either $I(v_1)$ would contain $I(v_k)$, or some vertex of P would have degree at least 3; either of the possibilities contradicts the assumptions of the lemma. □

Note that, for every induced subgraph H of G, the subfamily of $\mathcal{F}$ consisting of the intervals corresponding to the vertices of H forms an interval representation of H. We call such a representation *induced.* We call an induced path of G *chain-like* if it possesses the properties stated in Lemma 2 with respect to the induced interval representation.

Lemma 3. *Let C be a circuit without chords and let $\{I(v) : v \in V(C)\}$ be an interval representation of C. Then C has two adjacent vertices v_1 and v_2 such that the path $C - v_1 - v_2$ is chain-like.*

Proof. Since the case of triangle is trivial, we may assume that the length of C is at least 4. Let v_1 be the vertex satisfying $h(v_1) \leq h(v)$ for all $v \in V(C)$. Let x and y be the vertices adjacent to v_1. Consider the intervals $I(x)$ and $I(y)$. Since C has no chords and $|V(C)| \geq 4$, one of them, say $I(y)$, contains the other: $I(y) \supset I(x)$. Set $v_2 = y$. Let z be the vertex that is adjacent to v_2 and distinct from v_1. Consider the path $P = C - v_1 - v_2$. Note that x and z are the endpoints of P. We can have the two cases: either $I(z) \ni h(v_2)$ (Case 1, see Figure 2) or $I(z) \ni t(v_2)$ (Case 2, see Figure 3). Notice that $I(z) \subset I(v_1)$ in Case 1 and $I(z) \cap I(v_1) = \emptyset$ in Case 2. One can easily check that in both cases if at least one of the intervals $I(x)$ and $I(z)$ were contained in another interval of the induced representation of P then either one of the vertices of C would have degree at least 3, or $h(v) < h(v_1)$ for some $v \in V(C)$, contradicting the choice of v_1. Thus the path P satisfies the assumptions of Lemma 2 and therefore is chain-like. $\square$

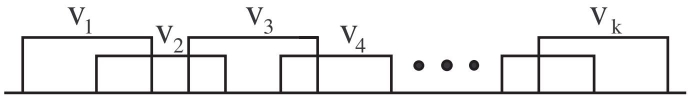

Figure 1.

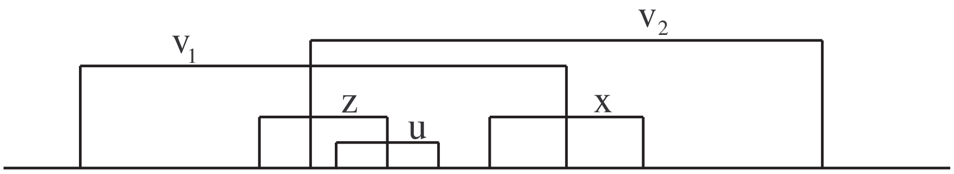

Figure 2.

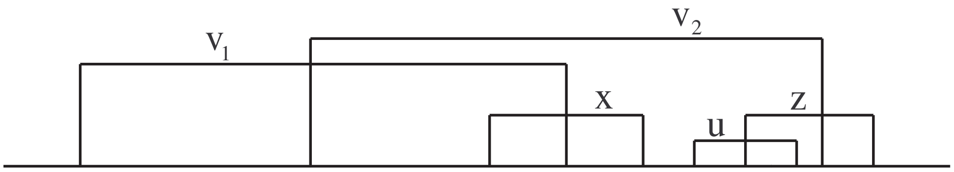

Figure 3.

Proof of Theorem. Let G be a circle graph of girth at least 5 and $\{I(v) : v \in V(G)\}$ be an interval representation of G. Assume that the theorem is false and G is a minimum counterexample. Then G is connected and has minimum degree at least 3. It follows that G has a circuit. Let C be a circuit of minimum length among all circuits of G with minimum $\beta(C)$. Observe that C has no chords for otherwise G would contain a shorter circuit C' with $V(C') \subset V(C)$ and, consequently, $\beta(C') \leq \beta(C)$. Then, according to the proof of Lemma 3, the induced representation of C can be either of the two different types — Case 1 and Case 2 — displayed in Figures 2 and 3 respectively. In the argument below we will treat the both cases simultaneously.

Invoke notation for the vertices of C from the proof of Lemma 3. Let u be the vertex of C adjacent to z and distinct from x. Then, since G has no cycles of length ≤ 4, $I(u) \subset I(v_1) \cap I(v_2)$ in Case 1 (see Figure 2) and $I(u) \subset I(v_2) \setminus I(v_1)$ in Case 2 (see Figure 3).

Claim 1. *Each vertex of G not lying on C has at most one neighbour on C.*

Assume to the contrary that some vertex $w \notin V(C)$ has two neighbours w_1 and w_2 on C. Call the two circuits passing through w_1, w, w_2 and a segment of C *splitting.* Recall that the splitting circuits have length at least 5 and, consequently, either of them is shorter than C. Consider Case 1. Then $I(w)$ contains neither $I(v_1)$ nor $I(v_2)$, for otherwise a splitting circuit would have length 4. Since G has no triangles, $I(w)$ is contained in $I(v_i)$ for some $i \in \{1, 2\}$. However then the base of one of the splitting circuits is contained in the base of C. Thus, taking into account that the splitting circuits are shorter than C, we arrive at a contradiction with the choice of C. Consider now Case 2. Here $I(w) \not\supset I(v_2)$ for otherwise, again, a splitting circuit would have length 4. If $I(w)$ overlapped $I(v_2)$, then one of the splitting circuits, being shorter than C, would have the same base, contradicting the choice of C. Thus $I(w) \subset I(v_2)$. However then the base of C (which coincides with $I(v_2)$) properly contains the base of one of the splitting circuits, a contradiction.

Claim 2. *u is a cut vertex of G.*

Suppose not. Then, since the degree of u is at least 3, G has a path $P = us_1 \ldots s_l w$ from u to some different vertex w on C whose inner vertices does not lie on C. Let the path P have minimum length among all such paths. Since G has no triangles, either $I(s_1) \supset I(u')$, or $I(s_1) \subset I(u')$ where u' is a neighbour of u on C. Assume that the former case holds. Then, as G has no triangles, $u' \neq z$. Further, by Claim 1, $I(s_1) \subset I(v_2)$. Hence $I(s_1)$ overlaps the union of the intervals corresponding to the vertices of the subpath of $C - v_1 - v_2$ from u' to x and consequently, by Lemma 1, overlaps $I(v)$ for some v on that path. However then s_1 has more than one neighbour on C, which is impossible by Claim 1. Thus, $I(s_1) \subset I(u')$. From this, using Lemma 1 and the assumption that P has minimum length, we obtain that either $I(s_k) \subset I(u')$ for all k, or $w = u'$. Since, by Lemma 2, the path $C - v_1 - v_2$ is chain-like, we conclude that in the former case w must be the other (distinct from u) neighbour of u' on C. So, in any case the distance between u and w along C is at most two. However then the base of the circuit consisting of P and the shortest segment of C between u and w, is properly contained in the base of C, a contradiction.

Let H' be the component of the graph $G - u$ that does not contain C. By Lemma 1, for each vertex $v \in V(H')$, we have $I(v) \subset I(v_1) \cap I(v_2)$ in Case 1 and $I(v) \subset I(v_2)$ in Case 2. Consider the induced subgraph $H = H' \cup u$. By above if H had a circuit C' then the base of C' would be a proper subinterval of the base of C, which is impossible. Consequently, H is a tree and, as such, has a pending vertex which is also pending in G, contradicting the assumption that G is a counterexample. $\square$

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