A Characterization of Seymour Graphs

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Abstract

Following Gerards [1] we say that a connected undirected graph G is a *Seymour graph* if the maximum number of edge disjoint T-cuts is equal to the cardinality of a minimum T-join for every even subset T of $V(G)$. Several families of graphs have been shown to be subfamilies of Seymour graphs (Seymour [4][5], Gerards [1], Szigeti [6]). In this paper we prove a characterization of Seymour graphs which was conjectured by Sebő and implies the results mentioned above.

1 Introduction

T. Graphs in this paper are undirected, connected, and may have loops and multiple edges.

Let G be a graph. For $F \subseteq E(G)$ and $x, y \in V(G)$, we write $xy \in F$ if some edge of G with endpoints x and y is in F.

For $X \subseteq V(G)$, the *cut* $\delta(X)$ is the set of edges connecting X and $V(G) \setminus X$, $N(X) = \{v \in V(G) \setminus X : v \text{ has neighbors in } X\}$. If $X = \{x\}$ we write $\delta(x)$, $N(x)$. For $F \subseteq E(G)$ and $v \in V(G)$, let $d_F(v)$ denote the degree of v in the subgraph of G spanned by F. A pair (G, T) where T is an even subset of $V(G)$ is called a *graft.*

Let T be an even subset of $V(G)$. If $|X \cap T|$ is odd the cut $\delta(X)$ is called a *T-cut.* A set F of edges of G is a *T-join* if $T = \{v \in V(G) \mid d_F(v) \text{ is odd}\}$. Let $\nu(G, T)$ denote the maximum number of edge disjoint T-cuts and $\tau(G, T)$ the cardinality of a minimum T-join in G.

Since every T-join meets every T-cut,

$$\nu(G, T) \leq \tau(G, T). \tag{1}$$

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The simplest example of a graft for which (1) holds with strict inequality is $(K_4, V(K_4))$. However, several families of graphs satisfying (1) with equality for every even vertex subset T have been found. These are bipartite graphs (Seymour [5]), series-parallel graphs (Seymour [4]), graphs containing neither an odd K_4 nor an odd prism (Gerards [1]). The last family contains bipartite and series-parallel graphs.

Following Gerards [1], we call a graph G a *Seymour graph* if (1) holds with equality for all even subsets $T \subseteq V(G)$.

Recently Szigeti [6] refined the Gerards' result having proved that if a graph is not a Seymour graph than it should contain either an odd K_4 or an odd prism which are not Seymour graphs. Sebő (unpublished) conjectured a necessary and sufficient condition for a graph to be a Seymour graph. His conjecture is stated in terms of conservative weightings and implies the Szigeti's result (and whence other results mentioned above).

Let G be a graph and $\mathbf{w} : E(G) \to \{-1, +1\}$. The weighting $\mathbf{w}$ is called *conservative* if G has no cycle of negative total weight (a loop is considered as a cycle). For any $F \subseteq E(G)$, the weighting $\mathbf{w}_F$ is defined by the equation:

$$\mathbf{w}_F(e) := \begin{cases} -1 & \text{if } e \in F, \\ +1 & \text{if } e \notin F. \end{cases}$$

The following observation (Mei Gu Guan's lemma [3]) reveals a one-to-one correspondence between conservative weightings and T-joins of minimum cardinality:

a T-join F is of minimum cardinality if and only if $\mathbf{w}_F$ is a conservative weighting.

A *conservative graph* is a pair $(G, \mathbf{w})$ in which G is a graph, $\mathbf{w}$ is a conservative weighting of G. For any conservative graph $(G, \mathbf{w})$, denote $E^-(\mathbf{w}) = \{e \in E(G) : \mathbf{w}(e) = -1\}$ and $T(\mathbf{w}) = \{v \in V(G) : d_{E^-(\mathbf{w})}(v) \text{ is odd}\}$. From now on we assume that $E^-(\mathbf{w}) \neq \emptyset$.

Given a conservative graph $(G, \mathbf{w})$, a cycle C of G is called a *$0(\mathbf{w})$-cycle* if the total weight of the edges of C is equal to zero. A graph G is an *odd K_4* if it is a subdivision of K_4 such that each cycle bounding a face of G has an odd length. A graph G is an *odd prism* if it is a subdivision of triangular prism such that each cycle bounding a triangular face of G has an odd length while each cycle bounding a quadrangular face has an even length.

Conjecture (Sebő). A graph G is not a Seymour graph if and only if there exist a conservative weighting $\mathbf{w}$ and $0(\mathbf{w})$-cycles C_1, C_2 such that the graph $C_1 \cup C_2$ is either an odd K_4 or an odd prism.

The conjecture implies that the class of Seymour graphs belongs to co-NP. The 'if' part of the conjecture was shown to be true by Sebő (unpublished). Section 3 of this paper presents a proof of the 'only if' part modulo Lemma 1. In the next section we state basic known results needed in the proof of the conjecture. Sections 4 – 5 are devoted to the proof of Lemma 1.

2 Background

In this section we present basic known results to which we will refer further. To do this some extra notation and definitions are needed.

A graph G is called *1-extendable* if each edge of G lies in a perfect matching and *factor-critical* if $G - x$ has a perfect matching for each $x \in V(G)$. A subdivision of a graph G is said to be *even* if the number of new vertices inserted in every edge of G is even. Clearly, any even subdivision of K_4 (respectively, of triangular prism) is an odd K_4 (respectively, an odd prism).

The first result is an easy consequence of Theorem 5.4.11 in [2].

Theorem 1 (Lovász) *Let G be a 1-extendable non-bipartite graph. Then G contains an even subdivision of either K_4 or triangular prism.*

$\square$

Let $(G, \mathbf{w})$ be a conservative graph. For any $x, y \in V(G)$, denote by $\lambda_{\mathbf{w}}(x, y)$ the length of a shortest path connecting x and y with respect to the edge length function $\mathbf{w}$. For $x \in V(G)$, let $m = m(x) = \min\{\lambda_{\mathbf{w}}(x, v) : v \in V(G)\}$, $M = M(x) = \max\{\lambda_{\mathbf{w}}(x, v) : v \in V(G)\}$, $V^i = V^i(x) = \{v \in V(G) : \lambda_{\mathbf{w}}(x, v) = i\}$, $G^i = G^i(x) = G[\bigcup_{j=m}^{i} V^j]$, $\tilde{G}^i = G^i - E(G[V^i])$. Let further $\mathcal{D}^i = \mathcal{D}^i(x)$ be the collection of vertex sets of components of G^i and $\mathcal{Q}^i = \mathcal{Q}^i(x)$ be the collection of vertex sets of components of $\tilde{G}^i$. Set $\mathcal{D} = \mathcal{D}(x) = \bigcup_{i=m}^{M} \mathcal{D}^i$, $\mathcal{Q} = \mathcal{Q}(x) = \bigcup_{i=m}^{M} \mathcal{Q}^i$, and $\mathcal{R} = \mathcal{R}(x) = \mathcal{D} \cup \mathcal{Q}$.

The assertions of the second theorem are special cases of Theorem 4.4 and Lemma 5.7 in [3].

Theorem 2 (Sebő) *Let $(G, \mathbf{w})$ be a conservative graph and $x \in V(G)$. Then*

(s1) *if $vu \in E^-(\mathbf{w})$, $\lambda_{\mathbf{w}}(x, v) = \lambda_{\mathbf{w}}(x, u) = i$, then v and u are in different elements of $\mathcal{Q}^i$ (i.e. in different components of $\tilde{G}^i$);*

(s2) $|\delta(R) \cap E^-(\mathbf{w})| = 1$ *if* $x \notin R \in \mathcal{R}(x)$,
$|\delta(R) \cap E^-(\mathbf{w})| = 0$ *if* $x \in R \in \mathcal{R}(x)$;

Moreover, let $m = m(x) < 0$, $D \in \mathcal{D}^m(x)$, and $\mathbf{w}'$ be the weighting $\mathbf{w}|_{G[D]}$. If $r \in D$ is the endvertex of the edge in $\delta(D) \cap E^-(\mathbf{w})$ (which is unique by (s2)), then

(s3) *for any $v \in D$, $\lambda_{\mathbf{w}'}(r, v) = 0$.*

$\square$

Let G be a graph and $\mathcal{P} = \{X_1, \ldots, X_k\}$ be a partition of $V(G)$. Denote by $G\langle\mathcal{P}\rangle$ the graph with the vertex set $\mathcal{P}$ and the edge set $E(G)$, which is obtained from G by shrinking every subset $X \in \mathcal{P}$ into a single vertex. For $X \subseteq V(G)$, denote by $\pi(X)$ the partition of $V(G)$ consisting of X and $|V(G) \setminus X|$ singletons.

We will use the following easy consequence of Theorem 2.

Corollary 1 *Suppose that the assumptions of Theorem 2 hold. Then*

(l1) $E^-(\mathbf{w})$ *forms in* $G[D \setminus \{r\}]$ *a perfect matching;*

(l2) $G[D]$ *is factor-critical;*

(l3) $(G\langle\pi(N_G(D))\rangle, \mathbf{w})$ *is a conservative graph.*

Proof. For any $v \in D$, since $\{v\} \in \mathcal{Q}^m$ and $m < 0$, it follows from (s2) that $|\delta(v) \cap E^-(\mathbf{w})| = 1$. Combining this with $|\delta(D) \cap E^-(\mathbf{w})| = 1$, we get (l1).

Using (l1) and statement (s3) of Theorem 2 we have that, for each $v \in D$, there exists an alternating (with respect to $E^- \cup E(G[D])$) path of even length connecting r with v in $G[D]$. Hence $G[D]$ is factor-critical.

Now let $A = N_G(D)$ and $G' = G\langle\pi(A)\rangle$. Note that by (s1) of Theorem 2 for each $R \in \mathcal{R}$, either $A \subseteq R$ or $A \cap R = \emptyset$. Hence shrinking A into a vertex would not change the set of cuts $\mathcal{M} = \{\delta(R) : R \in \mathcal{R}\}$. Any cut $M \in \mathcal{M}$ contains at most one edge of $E^-(\mathbf{w})$, each edge $e \in E(G')$ belongs to at most two members of $\mathcal{M}$ and each edge $e \in E^-(\mathbf{w})$ belongs to exactly two members of $\mathcal{M}$. Let us show that this provides the conservativeness of $(G', \mathbf{w})$. Consider the edge set C of a cycle in G'. Let $M_1, \ldots, M_k$ be the members of $\mathcal{M}$ meeting C. Then by above, $\sum_{i=1}^k |C \cap M_i \cap E^-(\mathbf{w})| \le k$, $\sum_{i=1}^k |C \cap M_i \cap E^-(\mathbf{w})| = 2|C \cap E^-(\mathbf{w})|$ and $\sum_{i=1}^k |C \cap M_i| \le 2|C|$. Since $\sum_{i=1}^k |C \cap M_i| \ge 2k$, we obtain $|C| \ge 2|C \cap E^-(\mathbf{w})|$. $\square$

3 The main result

Let $(G, \mathbf{w})$ be a conservative graph. Denote by $\Pi(G, \mathbf{w})$ the set of partitions $\mathcal{P}$ of $V(G)$ such that $(G\langle(\mathcal{P})\rangle, \mathbf{w})$ is conservative. Denote by $\Pi^*(G, \mathbf{w})$ the set of roughest partitions in $\Pi(G, \mathbf{w})$ (i.e. those which are not subpartitions of any member of $\Pi(G, \mathbf{w})$). Clearly, $\Pi(G, \mathbf{w}) \supseteq \Pi^*(G, \mathbf{w}) \neq \emptyset$ for any conservative graph $(G, \mathbf{w})$. We say that a partition $\mathcal{P}$ is *tree-like* if $G\langle\mathcal{P}\rangle$ has no cycles consisting of more than two edges. Note that if $\mathcal{P} \in \Pi^*(G, \mathbf{w})$ is tree-like then $E^-(\mathbf{w})$ induces a spanning tree of $G\langle(\mathcal{P})\rangle$. Moreover, it is clear that if $\Pi(G, \mathbf{w})$ contains a tree-like partition then so does $\Pi^*(G, \mathbf{w})$.

In the following theorem Sebő (private communication) proved (b)$\Longrightarrow$(c) and conjectured (c)$\Longrightarrow$(a) and (c)$\Longrightarrow$(b).

Theorem 3 *Let G be a undirected connected graph. The following conditions are equivalent:*

(a) *there exist a conservative weighting* $\mathbf{w}$ *and* $0(\mathbf{w})$*-cycles* C_1, C_2 *such that the graph* $C_1 \cup C_2$ *is either an odd* K_4 *or an odd prism;*

(b) *there exist a conservative weighting* $\mathbf{w}$ *and* $0(\mathbf{w})$*-cycles* C_1, C_2 *such that the graph* $C_1 \cup C_2$ *is non-bipartite;*

(c) G *is not a Seymour graph;*

(d) *there exists a conservative weighting* $\mathbf{w}$ *such that* $\Pi^*(G, \mathbf{w})$ *contains no tree-like partition.*

Proof. We shall prove (a)$\Longrightarrow$(b)$\Longrightarrow$(c)$\Longrightarrow$(d) $\Longrightarrow$(a).

(a)$\Longrightarrow$(b). Obvious.

(b)$\Longrightarrow$(c). Suppose to the contrary that G is a Seymour graph. Let C be an odd cycle of $H = C_1 \cup C_2$. Set $T = \{v \in V(G) : d_{E^-}(v) \text{ is odd}\}$, where $E^- = E^-(\mathbf{w})$. Let $\{D_1, \dots, D_\nu\}$ be a collection of edge disjoint T-cuts of G with $\nu = \nu(G, T)$. Since G is a Seymour graph, every D_i contains exactly one edge $e \in E^-$ but even number of edges from any cycle. Hence D_i has in common with each $0(\mathbf{w})$-cycle either no edges or exactly two (one from E^- and one from E^+). Consequently, $E(C) \subseteq \bigcup_{i=1}^{\nu} D_i$. But $|E(C) \cap D_i|$ is even for all $i = 1, \dots, \nu$. It follows that $|E(C)|$ is even, a contradiction.

(c)$\Longrightarrow$(d). Observe that if $G\langle\mathcal{P}\rangle$ is a multi-tree (i.e. can be obtained from a tree by multiplying edges and adding loops), then the T-cuts in G corresponding to the (multi)edges of $G\langle\mathcal{P}\rangle$ form a packing of size $|E_-|$.

(d)$\Longrightarrow$(a). The proof relies on the following lemma which is to be proved in the remainder of the paper.

Lemma 1 *Let* G *be a connected graph and* $X \subseteq V(G)$ *be a cut set of* G. *Suppose that* D *is the vertex set of a factor-critical component of* $G - X$ *such that* $N(D) = X$. *If* $G\langle\pi(X)\rangle - D$ *satisfies the condition (a) of Theorem 3 then so does* G.

Suppose that the implication does not hold and a graph G is a counterexample with the minimum number of vertices. That is

$$G \text{ satisfies (d) (with a weighting } \mathbf{w}) \text{ but does not satisfy (a)} \tag{2}$$

and

$$\text{(d)} \Longrightarrow \text{(a)} \quad \begin{array}{l} \text{for each graph } H \\ \text{with } |V(H)| < |V(G)|. \end{array} \tag{3}$$

Obviously,

$$|E^-(\mathbf{w})| \geq 2. \tag{4}$$

Claim. *Let Y be a nonempty subset of $V(G)$ such that $\pi(N_G(Y)) \in \Pi(G, \mathbf{w})$. If Y induces a factor-critical subgraph of G then $Y \cup N_G(Y) = V(G)$.*
Assume that $V(G) \setminus (Y \cup N_G(Y)) \neq \emptyset$ and Y has the minimum cardinality among all sets satisfying this property.

Denote $H = G[Y]$, $A = N_G(Y)$ and $G' = G\langle\pi(A)\rangle - Y$. Let $\mathbf{w}'$ be the weighting $\mathbf{w}|_{E(G')}$.

Note that G' does not satisfy (d) for otherwise by (3) it satisfies (a) and, consequently, by Lemma 1, so does G itself. Hence $\Pi^*(G', \mathbf{w}')$ contains a tree-like partition $\mathcal{P} = \{X_1, \ldots, X_k\}$.

If $E^-(\mathbf{w}') = \emptyset$ then for any $v \in V(G) \setminus (A \cup Y)$, $(G\langle\pi(N_G(v))\rangle, \mathbf{w})$ is conservative. By the minimality of $|Y|$, it follows that $|Y| = 1$ which contradicts (4).

Let $E^-(\mathbf{w}') \neq \emptyset$. In an pendant vertex X_i of $G'\langle\mathcal{P}\rangle$ not containing A, choose $v \in X_i$ incident to the edge $e \in E^- \cap \delta(X_i)$. Then $(G\langle\pi(N_G(v))\rangle, \mathbf{w})$ is conservative, and we have $|Y| = 1$ again. Hence $\{X_1, \ldots, X_k, Y\}$ is in $\Pi(G, \mathbf{w})$ and tree-like, a contradiction.

Let x be a vertex incident to an edge in $E^-(\mathbf{w})$. Consider $D \in \mathcal{D}^m(x)$, where $m = m(x)$. By Corollary 1 and Claim, $G[D]$ is factor-critical, $D \cup N_G(D) = V(G)$, and $E^-(\mathbf{w})$ forms a perfect matching in $G[\{x\} \cup D]$. Since x was chosen arbitrarily from $T(\mathbf{w})$, we conclude that $G[T(\mathbf{w})]$ is *bicritical*, i.e. for any $x, y \in V(G)$, the graph $G \setminus \{x, y\}$ has a perfect matching. It is easy to see that K_2 is the only bipartite bicritical graph. Thus, by (4) and by Theorem 1, G contains an even subdivision H of either K_4 or triangular prism. But the edges of H can be partitioned into three matchings M_1, M_2 and M_3 so that $M_1 \cup M_2$ and $M_1 \cup M_3$ are hamiltonian cycles in H. Thus G satisfies (a) with $\mathbf{w} = \mathbf{w}_{M_1}$, a contradiction.

□

4 Preliminary observations

In this section we state several easy observations to be referred to in the proof of Lemma 1.

Proposition 1 *Let G be a graph and $F \subseteq E(G)$. If F is a matching G then $\mathbf{w}_F$ is a conservative weighting.*

□

Proposition 2 *Let G be a graph and $\mathbf{w}$ be a weighting of G. The weighting $\mathbf{w}$ is conservative if and only if the weighting $\mathbf{w}|_{E(B)}$ is conservative for every block B of G.*

□

Proposition 3 *Let G be a graph. If $\mathbf{w}$ is a conservative weighting of $G\langle\pi(X)\rangle$ for some $X \subseteq V(G)$ then $\mathbf{w}$ is a conservative weighting of G.*

$\square$

Denote by $\mathcal{O}_1$ and $\mathcal{O}_2$ the sets of odd K_4-s and odd prisms respectively. Let $\mathcal{O}_1^e$ ($\mathcal{O}_2^e$) denote the even subdivisions of K_4 (prism, respectively). Clearly, $\mathcal{O}_k^e \subset \mathcal{O}_k$ $k = 1, 2$. Let $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ and $\mathcal{O}^e = \mathcal{O}_1^e \cup \mathcal{O}_2^e$.

Proposition 4 *If a graph G has a subgraph $H \in \mathcal{O}^e$ then G satisfies the condition (a) of Theorem 3.*

$\square$

Proposition 5 *Let $G \in \mathcal{O}^e$ and $f \in E(G)$. Then there exists a perfect matching M such that $f \in M$ and G is the union of two $0(\mathbf{w}_M)$-cycles. Moreover, if f is incident to a vertex of degree 3, then M and $0(\mathbf{w}_M)$-cycles can be chosen in such a way that f lies on both the $0(\mathbf{w}_M)$-cycles.*

$\square$

Proposition 6 *Let $G \in \mathcal{O}$ and let e_1, e_2 be adjacent edges of G. Then G has an even cycle passing through e_1 and e_2.*

$\square$

Proposition 7 *Let $G \in \mathcal{O}_2$ and let $u \in V(G)$ be a vertex of degree 3. Then G has three disjoint paths of the same parity connecting u with some vertex $v \in V(G)$. If, in addition, $G \in \mathcal{O}_2^e$ then G has three disjoint paths of odd length connecting u with some vertex $v \in V(G)$.*

$\square$

Proposition 8 *For any vertex v of degree at least three of a 1-extendable bipartite graph S, there exist three openly disjoint paths of odd length connecting v with some vertex $u \in V(S)$.*

$\square$

We prove this last statement. Let f_k $k = 1, 2, 3$ be three arbitrary edges incident to v and let F_k be three perfect matching of S containing f_k $k = 1, 2, 3$. Consider the alternating circuit D_k in $F_k \cup F_3$ containing f_k and f_3 $k = 1, 2$. Walking along on D_1 (starting from v on the edge f_1) let us denote by u the first vertex which is contained in D_2. (There exists such a vertex since the edge f_3 is contained by both D_1 and D_2.) We show that there exist three openly disjoint path connecting v and u. Let P_1 be that part of D_1 between v and u which contains f_1, and let P_2 and P_3 be the two parts of D_2 between v and u. Clearly, P_1, P_2 and P_3 are openly disjoint. Since G is bipartite, the parity of these paths are the same and since these paths are alternating for the matching F_3 they are odd paths. $\square$

5 Proof of Lemma 1

Let $\tilde{G} = G\langle\pi(X)\rangle - D$ and $G^* = G\langle\pi(V(G) \setminus X)\rangle - L$ and denote the contracted vertices by $\tilde{x}$ and x^*, respectively. Let $\mathbf{w}$ be a conservative weighting of $\tilde{G}$ and let $\tilde{C}_1$, $\tilde{C}_2$ be $0(\mathbf{w})$-cycles such that $\tilde{H} = \tilde{C}_1 \cup \tilde{C}_2 \in \mathcal{O}$. Denote by H (respectively, C_k, $k = 1, 2$) the subgraph of G spanned by the edges of $\tilde{H}$ (respectively, $\tilde{C}_k$). Let $X \cap V(H) = \{v_1, \ldots, v_l\}$. Since $l \leq 1$ implies $H = \tilde{H}$, we may assume that $l \in \{2, 3\}$. Let $f_k \in E(G)$, $k = 1, \ldots, l$, be an edge connecting v_k with some $u_i \in D$. Since $G[D]$ is factor-critical, G^* has perfect matchings $F_k \subset E(G^*)$ such that $f_k \in F_k$, $k = 1, \ldots, l$. Let $\tilde{S}$ be the component of the subgraph of G^* spanned by $\bigcup_{k=1}^{l} F_k$ which contains x^*. Let S be the subgraph of G spanned by $E(\tilde{S})$. The components of the union $F_k \cup F_j$, $1 \leq k, j \leq l$ are cycles and single edges and hence that containing x^* is a cycle. Thus, S is a connected subgraph of G

Case 1: $l = 2$.
Then $\tilde{S}$ is an even cycle so that S is an even path whose ends are v_1 and v_2. Let $M = F_1 \cap E(S)$. Set

$$\mathbf{w}^*(e) := \begin{cases} \mathbf{w}(e) & \text{if } e \in E(\tilde{G}), \\ -1 & \text{if } e \in M, \\ +1 & \text{otherwise.} \end{cases} \tag{5}$$

By Propositions 1 and 2, $\mathbf{w}^*$ is a conservative weighting of $G\langle\pi(X)\rangle$ and, consequently, by Proposition 3 it is that of G. Since S is an even path, $H \cup S$ belongs to $\mathcal{O}$. Note that $H \cup S = C_1 \cup C_2 \cup S$. We may have that either C_1, C_2 are both paths or exactly one of them, say C_1, is a path while C_2 is a cycle. If we have the former then $C_1 \cup S$ and $C_2 \cup S$ are the desired $0(\mathbf{w}^*)$-cycles, otherwise $C_1 \cup S$ and C_2 are those.

We assume next that $l = 3$. It follows that exactly one vertex v_i, say v_3, is incident to an edge which is contained in both cycles $\tilde{C}_1$ and $\tilde{C}_2$. In other words, $X \cap V(C_k) = \{v_k, v_3\}$, $k = 1, 2$.

Case 2: $l = 3$ *and* $\tilde{S}$ *is bipartite.*
By Proposition 8 S has three disjoint paths P_k, $k = 1, 2, 3$ of odd length connecting v_k with some $v \neq v_k$, $k = 1, 2, 3$. Let $Q = P_1 \cup P_2 \cup P_3$. The matching $M = F_3 \cap E(Q)$ covers all the vertices of Q except v_1 and v_2. Define $\mathbf{w}^*$ by the equation (5). Again, by Propositions 1, 2 $\mathbf{w}^*$ is a conservative weighting of $G\langle\pi(X)\rangle$, and whence by Proposition 3 it is that of G. Furthermore, we have that $H \cup Q$ belongs to $\mathcal{O}$ being the union of $0(\mathbf{w}^*)$-cycles $C_1 \cup P_1 \cup P_3$ and $C_2 \cup P_2 \cup P_3$.

Case 3: $l = 3$ *and* $\tilde{S}$ *is non-bipartite.*
Note that $\tilde{S}$ is 1-extendable. By Theorem 1 it follows that $\tilde{S}$ has a subgraph $\tilde{Q} \in \mathcal{O}^e$. Let Q be the subgraph of G spanned by $E(\tilde{Q})$. Note that $V(Q) \cap X \subseteq \{v_1, v_2, v_3\}$. If

$|V(Q)\cap X| \leq 1$ then $Q = \tilde{Q}$ and the conclusion obviously follows. If $|V(Q)\cap X| = 2$, we obtain the desired conclusion using Propositions 4, 6 and the argument of Case 1. Thus we may assume further that $V(Q)\cap X = \{v_1, v_2, v_3\}$. By Proposition 5 there exists a perfect matching M of $\tilde{Q}$ and $0(\mathbf{w_M})$-cycles $\tilde{D}_1$ and $\tilde{D}_2$ such that $f_3 \in M$, $f_3 \in E(\tilde{D}_1)\cap E(\tilde{D}_2)$ and $\tilde{Q} = \tilde{D}_1 \cup \tilde{D}_2$. Let D_k, $k = 1, 2$, be the subgraph of G spanned by $E(\tilde{D}_k)$. Note that $V(D_k)\cap X = \{v_k, v_3\}$, $k = 1, 2$.

Subcase 3.1: $\tilde{Q} \in \mathcal{O}_2^e$.
By Proposition 7 Q contains three disjoint paths P_k, $k = 1, 2, 3$, having odd length and such that P_k connects v_k with some $v \in V(Q)$, $v \neq v_k$, $k = 1, 2, 3$. It remains to apply the argument of Case 2 arriving at the same conclusion.

Subcase 3.2: $\tilde{Q} \in \mathcal{O}_1^e$, $\tilde{H} \in \mathcal{O}_2$.
By Proposition 7 H contains three disjoint paths P_k, $k = 1, 2, 3$, of the same parity and such that P_k connects v_k with some $v \in V(H)$, $v \neq v_k$, $k = 1, 2, 3$. If P_k-s are odd, the desired conclusion is obtained by repeating the argument of Case 2. If the parity of these paths is even, then $S \cup P_1 \cup P_2 \cup P_3 \in \mathcal{O}_1^e$ and we are done by Proposition 4.

Subcase 3.3: $\tilde{Q} \in \mathcal{O}_1^e$, $\tilde{H} \in \mathcal{O}_1$.
Note first that $H \cup Q$ is an odd prism. Define $\mathbf{w}^*$ by the equation (5). By Propositions 1, 2 $\mathbf{w}^*$ is a conservative weighting of $G\langle\pi(X)\rangle$ and thereby by Proposition 3 it is that of G. Finally, $H \cup Q$ is the union of $0(\mathbf{w}^*)$-cycles $D_1 \cup C_1$ and $D_2 \cup C_2$, as desired. $\square$

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