An 0.828-Approximation Algorithm for the Uncapacitated Facility Location Problem*

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Abstract

The uncapacitated facility location problem in the following formulation is considered:

$$\max_{S \subseteq I} \quad Z(S) = \sum_{j \in J} \max_{i \in S} b_{ij} - \sum_{i \in S} c_i,$$

where I and J are finite sets, and b_{ij}, $c_i \geq 0$ are rational numbers. Let Z^* denote the optimal value of the problem and $Z_R = \sum_{j \in J} \min_{i \in I} b_{ij} - \sum_{i \in I} c_i$. Cornuejols, Fisher and Nemhauser (1977) prove that for the problem with the additional cardinality constraint $|S| \leq K$, a simple greedy algorithm finds a feasible solution S such that $(Z(S) - Z_R)/(Z^* - Z_R) \geq 1 - e^{-1} \approx 0.632$. We suggest a polynomial-time approximation algorithm for the unconstrained version of the problem, based on the idea of randomized rounding due to Goemans and Williamson (1994). It is proved that the algorithm delivers a solution S such that $(Z(S) - Z_R)/(Z^* - Z_R) \geq 2(\sqrt{2} - 1) \approx 0.828$. We also show that there exists $\varepsilon > 0$ such that it is NP-hard to find an approximate solution S with $(Z(S) - Z_R)/(Z^* - Z_R) \geq 1 - \varepsilon$.

Keywords: facility location, satisfiability problem, approximation algorithm, performance guarantee

1 Introduction

In this paper we consider the maximization version of the uncapacitated facility location problem (UFLP) in the following classical formulation:

$$\max_{S \subseteq I} Z(S) = \sum_{j \in J} \max_{i \in S} b_{ij} - \sum_{i \in S} c_i, \tag{1}$$

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where $I = \{1, \dots, m\}$, $J = \{1, \dots, n\}$, and b_{ij}, $c_i \geq 0$ are rational numbers. The sets I and J are commonly interpreted as the set of possible sites of locating facilities and the set of clients respectively, c_i means the cost of opening facility at site i, b_{ij}, a profit from satisfaction of client j from facility i. UFLP is to maximize the total profit provided that all demands are satisfied (for applications and historical overview, see [5]). The objective function $Z(S)$ can take both positive and negative values and so there is a difficulty in the definition of measure of relative deviation for approximate solutions to (1). To overcome this Cornuejols, Fisher and Nemhauser [3] (see also [4] for a more extensive discussion on this topic) consider the problem with the shifted objective function $(Z(S) - Z_R)$ where $Z_R = \sum_{j \in J} \min_{i \in I} b_{ij} - \sum_{i \in I} c_i$ is a trivial lower bound for $Z(S)$. They prove that for (1) with the additional cardinality constraint $|S| \leq K$, a simple greedy algorithm finds a feasible solution S such that $(Z(S) - Z_R)/(Z^* - Z_R) \geq 1 - e^{-1} \approx 0.632$ where Z^* is an optimal value of the problem. We suggest a polynomial-time approximation algorithm for the unconstrained problem based on the idea of randomized rounding due to Goemans and Williamson (1994). It is proved that the algorithm delivers a feasible solution S such that $(Z(S) - Z_R)/(Z^* - Z_R) \geq 2(\sqrt{2} - 1) \approx 0.828$. This essentially improves on the bound of Cornuejols, Fisher and Nemhauser in the unconstrained case (when $K \geq m$). We also show that there exists $\varepsilon > 0$ such that it is NP-hard to find an approximate solution S with $(Z(S) - Z_R)/(Z^* - Z_R) \geq 1 - \varepsilon$. Our approach is based on the fact that UFLP in the shifted form

$$\max_{S \subseteq I} \; (Z(S) - Z_R) \tag{2}$$

admits a two-way approximation preserving reduction to a special case (MAX SAT*) of the maximum satisfiability problem. The peculiarities of MAX SAT* permit us to design approximation algorithms with better performance guarantees than those known for the general case.

2 Approximation preserving reductions between UFLP in the shifted form and MAX SAT*

Let $x_1, x_2, \dots, x_m$ be Boolean variables. A *literal* is either a variable x or its negation $\overline{x}$. A *clause* is a disjunction of literals taken from $\{x_1, x_2, \dots, x_m\}$. The Maximum Satisfiability Problem (MAX SAT) is formulated as follows: given a set of clauses $\mathcal{C}$ and a weight function $w : \mathcal{C} \to \mathbf{R}$, find an assignment of truth values to $x_1, x_2, \dots, x_m$ maximizing the total weight of satisfied clauses. For brevity we shall informally identify the truth and false values with the numbers 1 and 0 respectively.

Consider an instance of UFLP. For any $j \in J$, let $(i_1(j), i_2(j), \dots, i_m(j))$ be a permutation of I such that $b_{i_1(j),j} \geq b_{i_2(j),j} \geq \dots \geq b_{i_m(j),j}$. For any $k = 1, \dots, m$ and $j \in J$, let $C_{kj} = \{i_1(j), i_2(j), \dots, i_k(j)\}$ and $w_{kj} = b_{i_k(j),j} - b_{i_{k+1}(j),j}$, where $b_{i_{m+1}(j),j}$ is defined to be equal to $b_{i_m(j),j}$ for all j. Consider the instance of MAX SAT whose set of clauses is $\{\bigvee_{i \in C_{kj}} x_i : k = 1, \dots, m, j \in J\} \cup \{\overline{x}_i : i \in I\}$ and the weight function w is defined by $w(\bigvee_{i \in C_{kj}} x_i) = w_{kj}$, and $w(\overline{x}_i) = c_i$ for all $k = 1, \dots, m$, $i \in I$ and $j \in J$.

Take $S \subseteq I$ and let $x = (x_1, x_2, \ldots, x_m)$ be the incidence vector of S . Then

$$\begin{aligned}\sum_{j\in J}\max_{i\in S} b_{ij} - \sum_{i\in S} c_i &= \sum_{j\in J}\Big(\sum_{k=1}^{m} w_{kj}\big(\bigvee_{i\in C_{kj}} x_i\big) + \min_{i\in I} b_{ij}\Big) + \sum_{i\in I} c_i\overline{x}_i - \sum_{i\in I} c_i \\ &= \sum_{k=1}^{m}\sum_{j\in J} w_{kj}\big(\bigvee_{i\in C_{kj}} x_i\big) + \sum_{i\in I} c_i\overline{x}_i + \sum_{j\in J}\min_{i\in I} b_{ij} - \sum_{i\in I} c_i.\end{aligned}$$

It follows that

$$Z(S) - Z_R = \sum_{k=1}^{m}\sum_{j\in J} w_{kj}\big(\bigvee_{i\in C_{kj}} x_i\big) + \sum_{i\in I} c_i\overline{x}_i. \tag{3}$$

Thus, the value of $S \subseteq I$ in the instance (1) of UFLP is equal to the value of the corresponding truth assignment x in the above constructed instance of MAX SAT.

The equality (3) shows that UFLP reduces to the special case of MAX SAT that can be written in the following general form:

$$\max_{x\in\{0,1\}^m} \sum_{l=1}^{L} f_l\big(\bigvee_{i\in C_l} x_i\big) + \sum_{i=1}^{m} g_i\overline{x}_i, \tag{4}$$

where $C_l \subseteq \{1, \ldots, m\}$ and $f_l, g_i \geq 0$, $l = 1, \ldots, L$, $i = 1, \ldots, m$. We shall refer to the problem (4) as MAX SAT*.

The reduction from MAX SAT* to UFLP is as follows. Let (4) be an instance of MAX SAT*. Consider the instance (2) of UFLP with $I = \{1, \ldots, m\}$, $J = \{1, \ldots, L\}$, $c_i = g_i$, $i = 1, \ldots, m$, and

$$b_{ij} = \begin{cases} f_j & \text{if } i \in C_j, \\ 0 & \text{otherwise.}\end{cases}$$

Observe that if we apply to this instance of UFLP the reduction described previously we shall arrive at the original instance of MAX SAT*. Again, by the above, the value of a subset $S \subseteq I$ in the instance of UFLP is equal to the value of the incidence vector of S in the original instance of MAX SAT*. Thus, *the described reductions between UFLP in the shifted form and MAX SAT* preserve relative errors of the corresponding feasible solutions.*

3 A $2(\sqrt{2}-1)$-approximation algorithm for MAX SAT*

In this section we present an $2(\sqrt{2} - 1)$-approximation algorithm for solving MAX SAT*.

Following [6], we rewrite (4) as the integer program:

$$\max \sum_{l=1}^{L} f_l z_l + \sum_{i=1}^{m} g_i(1 - y_i)$$

$$\sum_{i \in C_l} y_i \geq z_l, \quad l = 1, \ldots, L, \tag{IP}$$

$$y_i \in \{0, 1\}, \quad i = 1, \ldots, m,$$

$$0 \leq z_l \leq 1, \quad l = 1, \ldots, L.$$

The relations $y_i = x_i$ and $z_l = \bigvee_{i \in C_l} x_i$, $i = 1, \ldots, m$, $l = 1, \ldots, L$, establish a one-to-one correspondence between optimal solutions of the two problems and display that the optimal value F^*_{IP} of (IP) is equal to the optimal value of MAX SAT*. Let (LP) stand for the linear relaxation of (IP) obtained by replacing the $y_i \in \{0, 1\}$ constraints with the $0 \leq y_i \leq 1$ constraints. Clearly, $F^*_{IP} \leq F^*_{LP}$.

Our algorithm is an ad hoc modification of the randomized $(1 - 1/e)$-approximation algorithm for MAX SAT described in [6] (Section 3). The algorithm consists of the two steps. At step 1 the algorithm finds an optimal solution (y^*_i, z^*_l) to the linear program (LP). At step 2 the algorithm independently sets for each i, $y_i := 1$ with probability $p_i = 1 - \lambda^* + \lambda^* y^*_i$ and $y := 0$ with probability $1 - p_i$ where $\lambda^* = 2(\sqrt{2} - 1)$. The deterministic algorithm is obtained from this by using the method of conditional probabilities (see [1]). Note that the algorithm just described differs from that of Goemans and Williamson only in step 2 at which their algorithm sets $p_i = y^*_i$.

To make transparent the underlying points we shall consider the general case with $p_i = 1 - \lambda + \lambda y_i$, $\lambda \in [0, 1]$, that includes the both algorithms. Let $W(\lambda)$ denote the expected value of the objective function of (IP) on the solution retrieved by the algorithm defined by some fixed λ. To prove the performance guarantee of the algorithm just as in [6] we shall compare $W(\lambda)$ with F^*_{LP}. The argument below follows the lines of the proof of Lemma 3.1 in [6].

Clearly,

$$\begin{aligned} W(\lambda) &= \sum_{l=1}^{L} f_l \big(1 - \prod_{i \in C_l} (1 - p_i)\big) + \sum_{i=1}^{m} g_i (1 - p_i) \\ &= \sum_{l=1}^{L} f_l \big(1 - \lambda^{|C_l|} \prod_{i \in C_l} (1 - y^*_i)\big) + \lambda \sum_{i=1}^{m} g_i (1 - y^*_i). \end{aligned} \tag{5}$$

We show that

$$1 - \lambda^k \prod_{i \in C_l} (1 - y^*_i) \geq \Delta_k(\lambda) z^*_l, \tag{6}$$

where $k = |C_l|$ and $\Delta_k(\lambda) = 1 - \lambda^k (1 - 1/k)^k$. Indeed, applying the arithmetic-geometric mean inequality and using the constraint on z_l we obtain that

$$1 - \lambda^k \prod_{i \in C_l} (1 - y^*_i) \geq 1 - \lambda^k \Big(1 - \frac{\sum_{i \in C_l} y^*_i}{k}\Big)^k \geq 1 - \lambda^k (1 - \frac{z^*_l}{k})^k .$$

Now (6) follows from the fact that the function $f(t) = 1 - \lambda^k (1 - t/k)^k$ is concave on the segment $[0, 1]$ (this can be easily verified by computing its second derivative) and $f(0) = 1 - \lambda^k$, $f(1) = \Delta_k(\lambda)$.

Thus, by (5) and (6)

$$W(\lambda) \geq \min\Big\{\min_{k\geq 1} \Delta_k(\lambda), \lambda\Big\} F^*_{LP} \geq \min\Big\{\min_{k\geq 1} \Delta_k(\lambda), \lambda\Big\} F^*_{IP} .$$

According to the method of conditional probabilities $\varphi(\lambda) = \min\Big\{\min_{k\geq 1} \Delta_k(\lambda), \lambda\Big\}$ is the performance guarantee of the derandomized algorithm. We now show that λ^* is the best possible value of the parameter λ in the algorithm, that is $\varphi(\lambda) \leq \varphi(\lambda^*)$ for all $\lambda \in [0,1]$. Observe first that the maximum value of $\min\{\Delta_2(\lambda), \lambda\}$ is attained at $\lambda = \lambda^* = 2(\sqrt{2}-1)$, which is a root of the quadratic equation $\Delta_2(\lambda) = \lambda$. So,

$$\min\{\Delta_2(\lambda), \lambda\} \leq \min\{\Delta_2(\lambda^*), \lambda^*\} = \lambda^*.$$

We claim next that $\Delta_k(\lambda^*) \geq \Delta_2(\lambda^*)$ for all $k \geq 1$. Indeed, for $k = 1, 3, 4$ this inequality can be verified by a straightforward computation. For all remaining $k \geq 5$ it is obtained from the inequalities

$$\Delta_k(\lambda^*) \geq 1 - \tfrac{\lambda^{*k}}{e} \geq 1 - \tfrac{\lambda^{*5}}{e} > 0.85 > \Delta_2(\lambda^*).$$

Thus

$$\varphi(\lambda) \leq \min\{\Delta_2(\lambda), \lambda\} \leq \min\{\Delta_2(\lambda^*), \lambda^*\} = \lambda^* = \varphi(\lambda^*).$$

4 Hardness of approximation

In this section we show that for some $\varepsilon > 0$ existence of $(1-\varepsilon)$-approximation algorithm for solving MAX SAT* implies NP = P. From this and the reduction described in the end of Section 2 it follows that MAX SAT* as well as UFLP in the form (2) admits no polynomial-time approximation scheme.

Let SAT denote the set of satisfiable formulas in the conventional (decision) satisfiability problem (which we shall also refer to as SAT). Let MAX 3SAT(5) stand for the special case of MAX SAT in which each clause has length 3 and unit weight, and each variable enters exactly 5 clauses.

For any instance I of MAX SAT, let $w^*(I)$ denote the total weight of clauses satisfied by an optimal assignment in I and let $r(I)$ be equal to $w^*(I)$ divided by the total weight of clauses in I. Our proof relies upon the following result.

Theorem 1 (Arora and Lund, see [2], Theorem 10.2). *There exists a polynomial time reduction τ from SAT to MAX 3SAT(5) that, for some fixed $\varepsilon > 0$ and all SAT-instances I, ensures:*

$$I \in SAT \Rightarrow r(\tau(I)) = 1,$$

$$I \notin SAT \Rightarrow r(\tau(I)) < 1 - \varepsilon. \quad \square$$

Note that, since $\frac{5m}{3}$ is the total number of clauses in MAX 3SAT(5), $w^*(\tau(I)) = \frac{5m}{3} r(\tau(I))$.

We now describe a polynomial-time reduction σ from MAX 3-SAT(5) to MAX SAT*. Let $\mathcal{C} = \{C_l : l = 1, \ldots, L\}$ $(L = 5m/3)$ be the set of clauses in an instance I of MAX 3-SAT(5). The instance $\sigma(I)$ of MAX SAT* is defined as follows. The set of variables of $\sigma(I)$ is $\{y_1, \ldots, y_m\} \cup \{z_1, \ldots, z_m\}$. The set of clauses is the union of the three sets $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{C}'$, where $\mathcal{D} = \{y_i \vee z_i : i = 1, \ldots, m\}$, $\mathcal{E} = \{\overline{y}_i : i = 1, \ldots, m\} \cup \{\overline{z}_i : i = 1, \ldots, m\}$, and $\mathcal{C}'$ consists of the clauses $\{C_l : l = 1, \ldots, L\}$ in which each variable x_i is replaced by y_i and its negation $\overline{x}_i$, by z_i. The clauses in $\mathcal{D}$ have weight 11, those in $\mathcal{E}$ have weight 5, and those in $\mathcal{C}'$ have weight 1.

Proposition 1. *If (y^*, z^*) is an optimal assignment of $\sigma(I)$, then $y_i^* + z_i^* = 1$ for all $i = 1, \ldots, m$.*

Proof. Assume that $y_i^* + z_i^* = 0$ for some i. Replacing $y_i^* = 0$ by 1 in the assignment (y^*, z^*) increases its weight at least by 1, which is impossible in view of optimality of (y^*, z^*). Assume now that $y_i^* + z_i^* = 2$ for some i. Recall that the variables y_i^*, z_i^* totally enter exactly 5 clauses in C_l. Then at least one of them, say y_i^*, enters less then 5 clauses in C_l. Again, replacing $y_i^* = 1$ by 0 in the assignment (y^*, z^*) increases its weight at least by 1, which is a contradiction to optimality of (y^*, z^*). □

The following is an obvious consequence of Proposition 1 and the above description of the reduction σ.

Proposition 2. *The vector x^* is an optimal assignment of an instance I of MAX 3SAT(5) if and only if (y^*, z^*), where $y_i^* = x_i^*$, $z_i^* = 1 - x_i^*$, $i = 1, \ldots, m$, is an optimal assignment of $\sigma(I)$. Moreover, $w^*(I) = w^*(\sigma(I)) - 16m$.* □

Consider the polynomial-time reduction τ' from SAT to MAX SAT* that is the result of consecutive application of the reductions τ and σ. Using Theorem 1 and Proposition 2 we obtain the following. If $I \in$ SAT, then $w^*(\tau'(I)) = \frac{5m}{3} + 16m$ and since the total weight of clauses in $\tau'(I)$ is $\frac{5m}{3} + 16m$, $r(\tau'(I)) = 1$. Let $I \notin$ SAT. Then $w^*(\tau'(I)) = w^*(\tau(I)) + 16m = \frac{5m}{3} r(\tau(I)) + 16m < \frac{5m}{3}(1 - \varepsilon) + 16m$. Consequently,

$$r(\tau'(I)) < \frac{\frac{5m}{3}(1-\varepsilon) + 16m}{\frac{5m}{3} + 16m} = \frac{53 - 5\varepsilon}{53} = 1 - \tfrac{5}{53}\varepsilon.$$

Therefore, for any $\varepsilon' \leq \frac{5}{53}\varepsilon$, existence of a polynomial-time $(1 - \varepsilon')$-approximation algorithm for solving MAX SAT* would imply P=NP, as desired.

5 An extreme example

In this section we present an infinite series of instances of MAX SAT* with F_{IP}/F_{LP} tending to $2(\sqrt{2} - 1)$ as the size of the instance tends to ∞. This means that the performance guarantee of $2(\sqrt{2} - 1)$ is tight and, moreover, the rounding described in Section 3 is best possible in a sense among all roundings of an optimal solution to the linear relaxation (LP).

Consider the following instance of (IP): $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ where $\mathcal{C}' = \{\bar{x}_i : 1 \leq i \leq m\}$, $\mathcal{C}'' = \{x_i \vee x_j : 1 \leq i < j \leq m\}$, and $w(\bar{x}_i) = 1$, $w(x_i \vee x_j) = \beta/(m-1)$ where $\beta \geq 1$ and $1 \leq i \leq j \leq m$. Note that $|\mathcal{C}'| = m$, $|\mathcal{C}''| = m(m-1)/2$.

Our purpose below is to overestimate F_{IP}^*/F_{LP}^*.

Since $(1/2, \ldots, 1/2)$ is a feasible solution of (LP), we have

$$F_{LP}^* \geq \frac{\beta}{m-1} \cdot \frac{m(m-1)}{2} + \frac{m}{2} = m\frac{\beta + 1}{2}. \tag{7}$$

Let $L_m = \{l \in [0,1] : lm \text{ is integer}\}$. For any $l \in L_m$, let F_{IP}^l denote the total weight of the satisfied clauses in $\mathcal{C}''$ provided that exactly lm variables have value false. Then, using the symmetry of the instance, we obtain that

$$\begin{aligned}
F_{IP}^* &= \max_{l \in L_m} F_{IP}^l \\
&\leq \max_{l\in[0,1]} \Big\{ \Big(\frac{m(m-1)}{2} - \frac{lm(lm-1)}{2} \Big) \frac{\beta}{m-1} + lm \Big\} \\
&= \max_{l\in[0,1]} \Big\{ \frac{\beta m}{2} + lm - \frac{\beta lm(lm-1)}{2(m-1)} \Big\} \\
&\quad \Big(\text{since } \frac{m(lm-1)}{m-1} \geq lm - \frac{m}{m-1}\Big) \\
&\leq \max_{l\in[0,1]} \Big\{ m\big(\frac{\beta}{2} + l - \frac{\beta l^2}{2} + \frac{\beta l}{2(m-1)}\big) \Big\} \\
&\leq \max_{l\in[0,1]} \Big\{ m\big(\frac{\beta}{2} + l - \frac{\beta l^2}{2} + \frac{\beta}{2(m-1)}\big) \Big\} \\
&\quad \Big(\text{since } \beta \geq 1, \text{ the maximum is attained at } l = 1/\beta\Big) \\
&= m\Big(\frac{\beta^2+1}{2\beta} + \frac{\beta}{2(m-1)} \Big). \qquad (8)
\end{aligned}$$

Now set $\beta = 1+\sqrt{2}$. Then, by (7) and (8),

$$\begin{aligned}
\frac{F_{IP}^*}{F_{LP}^*} &\leq \frac{\beta^2+1}{\beta(\beta+1)} + \frac{\beta}{(m-1)(\beta+1)} \\
&= \frac{2(2+\sqrt{2})}{(1+\sqrt{2})(2+\sqrt{2})} + \frac{\beta}{(m-1)(\beta+1)} \\
&= 2(\sqrt{2}-1) + \frac{\beta}{(m-1)(\beta+1)} \quad \to \quad 2(\sqrt{2}-1),
\end{aligned}$$

as m tends to ∞.

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