A Well-Behaved Extension of the Vertex Covering Problem

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Abstract

The paper shows that several well-known results on properties of optimal solutions of the minimum weight vertex covering problem (Balinski [1], Balinski and Spielberg [2], Nemhauser and Trotter [8], Hammer et al.[7], Bourjolly et al.[3]) remain true for an extension of it.

1 Introduction

Let $G = (V, E)$ be a undirected graph with vertex set V and edge set E. A subset $X \subseteq V$ is called a *vertex covering* if every edge of G has at least one endpoint in X. The minimum weight vertex covering problem (VCP) is, given a graph G with positive vertex weights c_i $(i \in V)$, to find a vertex covering X^* with the minimum weight $\sum_{i \in X^*} c_i$. Clearly, a set $X \subseteq V$ is a vertex covering if and only if its complement $V \setminus X$ consists of pairwise nonadjacent vertices, i.e., is *stable*. So finding a minimum weight vertex covering is equivalent to finding a maximum weight stable set. Both problems are classic in discrete optimization and have been extensively investigated for recent decades. Despite NP-hardness (see Garey and Johnson [5]) they enjoy a number of remarkable properties which can be helpful when constructing efficient algorithms. To state some well-known results of this kind we have to reformulate the VCP as the following integer program:

$$\min \quad \sum_{i \in V} c_i x_i \ , \tag{1}$$

$$\begin{aligned} \text{s.t.} \qquad x_i + x_j \ &\geq \ 1, \ \ e = (i, j) \in E, &(2)\\ x_i \ &\geq \ 0, \ i \in V \ , &(3)\\ x_i \ &\in \ \{0, 1\}, \ i \in V \ . &(4) \end{aligned}$$

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The linear relaxation (1)–(3) is called the *fractional vertex covering problem* (FVCP). By the integrality gap of an integer program we shall mean the difference between its optimum value and that of its linear relaxation. The following theorem is a collection of known results most of which were originally stated for the maximum weight stable problem.

Theorem 1

(a1) (Balinski [1][2], Nemhauser and Trotter [8]) *There exists an optimal solution x to the FVCP (1)–(3) such that*

$$x_i \in \{0, 1/2, 1\},\ i \in V. \tag{5}$$

Such a solution can be found in polynomial time.

(a2) (Nemhauser and Trotter [8]) *Let x be an optimal solution to the* FVCP *(1)–(3). Let $I_p = \{i : x_i = p\}$, $p = 0, 1$. Then there exists an optimal solution x^* to the* VCP *(1)–(4) such that $x_i^* = p$ whenever $p \in \{0, 1\}$ and $i \in I_p$.*

(a3) (Hammer, Hansen and Simeone [7]) *If $x_i = p$ ($p \in \{0, 1\}$) for each optimal solution x to the* FVCP *(1)–(3), then $x_i^* = p$ for each optimal solution x^* to the* VCP *(1)–(4).*

(a4) (Bourjolly, Hammer and Simeone [3]) *There exists an $O(|V|^3)$ algorithm to decide whether the integrality gap of the VCP (1)–(4) is equal to zero and if so to find a minimum weight vertex covering.*

Remark 1. The problem (1), (2), (5) can be solved in $O(|V|^2|E|^{1/2})$ time. It reduces to the VCP on a bipartite graph with $2|V|$ vertices and $2|E|$ edges (see Nemhauser and Trotter [8]). The latter reduces to the maximum flow problem on the same graph and hence can be solved in $O(|V|^2|E|^{1/2})$ time (see Goldberg and Tarjan [6] and Cheriyan and Maheshwary [4]).

Suppose now that we are given a undirected graph $G = (V, E)$ with positive weights d_e on its edges, a finite set T with positive weights c_t on its elements, and a collection of nonempty vertex subsets $\alpha_t \subseteq V$ indexed by the elements of T. Consider the following integer program:

$$\min \quad \mathcal{F}(x, y, z) = \sum_{t \in T} c_t y_t + \sum_{e \in E} d_e z_e\ , \tag{6}$$

$$\begin{aligned}
\text{s.t.} \quad x_i + x_j + z_e &\geq 1, \quad e = (i, j) \in E, && (7)\\
x_i &\leq y_t, \quad i \in \alpha_t,\ t \in T, && (8)\\
x_i, y_t, z_e &\geq 0, \quad i \in V,\ t \in T,\ e \in E, && (9)\\
x_i, y_t, z_e &\in \{0, 1\}, \quad i \in V,\ t \in T,\ e \in E, && (10)
\end{aligned}$$

We shall refer to (6)–(10) as the *vertex subset choice problem* (VSCP). It is easy to see that the VSCP becomes equivalent to the VCP when $T = V$, $\alpha_i = \{i\}$, $i \in V$, $d_e > \sum_{i \in V} c_i$, $e \in E$. This observation shows that the VSCP may be viewed as an extension of the VCP.

Moreover, since for each optimal solution (x, y, z),

$$y_t = \max_{i \in \alpha_t} x_i \tag{11}$$

we can rewrite (6)–(10) in the following equivalent way:

$$\min \quad \sum_{t \in T} c_t \max_{i \in \alpha_t} x_i + \sum_{e \in E} d_e z_e \ ,$$

subject to (7), (10). Thus the VSCP essentially extends the VCP at least in the two aspects: its objective function is nonlinear and, moreover, contains "penalties" d_e for not covering edges of G.

The main result of this paper is that the properties (a1)–(a4) of VCP stated in Theorem 1 remain true for the VSCP.

Theorem 2

(b1) *There exists an optimal solution (x, y, z) to the linear relaxation (6)-(9) satisfying the property*

$$x_i,\ y_t,\ z_e \in \{0, 1/2, 1\},\ i \in V,\ t \in T,\ e \in E. \tag{12}$$

Such a solution can be found in $O(\max\{|V||T|, |E|\}^{2.5})$ time.

(b2) *Let (x, y, z) be an optimal solution to the linear relaxation (6)–(9). Let $I_p = \{i : x_i = p\}$, $p = 0, 1$. Then there exists an optimal solution (x^*, y^*, z^*) to the* VSCP *(6)–(10) such that $x_i^* = p$ whenever $p \in \{0, 1\}$ and $i \in I_p$.*

(b3) *If $x_i = p$ $(p \in \{0, 1\})$ for each optimal solution (x, y, z) to the linear relaxation (6)–(9), then $x_i^* = p$ for each optimal solution (x^*, y^*, z^*) to the* VSCP *(6)–(10).*

(b4) *There exists an $O(\max\{|V||T|, |E|\}^3)$ algorithm to decide whether the integrality gap of (6)-(10) is equal to zero and if so to find an optimal solution.*

A proof of Theorem 2 is contained in Section 3 and uses the results (a1)–(a4) stated in Theorem 1 and an appropriate transformation of the VSCP into the VCP. This transformation is described in Section 2.

2 A transformation of the VSCP into the VCP

We begin with converting G into a directed graph $\vec{G} = (V, \vec{E})$ by arbitrarily orienting its edges. For any $e \in \vec{E}$, denote by $h(e)$ and $t(e)$ the head and the tail of e, respectively. With the instance (6)–(10) of the VSCP associate the following integer program:

$$\begin{aligned}
\mathcal{H}(u,v,w,q',q'') = \quad \min \quad & \sum_{t\in T} A_t v_t + \sum_{i\in V} B_i u_i + \sum_{e\in\vec{E}} d_e(w_e^{h(e)} + w_e^{t(e)}) \\
& + \Delta \sum_{t\in T}\sum_{i\in\alpha_t} (q'_{it} + q''_{it}) , \qquad (13)
\end{aligned}$$

$$\begin{aligned}
\text{s.t.} \qquad u_i + w_e^i &\geq 1, \ \ e \in \vec{E}, i \in \{h(e), t(e)\} && (14)\\
u_i + q'_{it} &\geq 1, \ \ i \in \alpha_t, \ \ t \in T, && (15)\\
v_t + q''_{it} &\geq 1, \ \ i \in \alpha_t, \ \ t \in T, && (16)\\
w_e^{h(e)} + w_e^{t(e)} &\geq 1, \ \ e \in \vec{E}, && (17)\\
q'_{it} + q''_{it} &\geq 1, \ \ i \in \alpha_t, \ \ t \in T, && (18)\\
u_i, v_t, w_e^j, w_e^k, q'_{it}, q''_{it} &\geq 0, \ i \in V, \ t \in T, \ e = (j,k) \in \vec{E}, && (19)\\
u_i, v_t, w_e^j, w_e^k, q'_{it}, q''_{it} &\in \{0,1\}, \ i \in V, \ t \in T, \ e = (j,k) \in \vec{E}, && (20)
\end{aligned}$$

where $A_t = \Delta|\alpha_t| - c_t$, $B_i = \Delta|\{t : i \in \alpha_t\}|$, and $\Delta > \max\{c_t : t \in T\}$.

It is easy to see that (13)-(20) forms an instance of the VCP.

Denote by φ_0 the mapping

$$\varphi_0 : (u, v, w, q', q'') \to (x, y, z)$$

defined by the equalities

$$\begin{aligned}
x_i &= u_i, \quad i \in V, && (21)\\
y_t &= 1 - v_t, \quad t \in T, && (22)\\
z_e &= w_e^{h(e)} + w_e^{t(e)} - 1 , \quad e \in \vec{E}. && (23)
\end{aligned}$$

Next, denote by ψ the mapping

$$\psi : (x, y, z) \to (u, v, w, q', q'')$$

defined by the equalities

$$\begin{aligned}
u_i &= x_i, \ \ i \in V, && (24)\\
v_t &= 1 - y_t, \ \ t \in T, && (25)\\
w_e^{h(e)} &= 1 - x_{h(e)} , e \in \vec{E}, && (26)\\
w_e^{t(e)} &= x_{h(e)} + z_e , e \in \vec{E}, && (27)\\
q'_{it} &= 1 - x_i , && (28)\\
q''_{it} &= y_t . && (29)
\end{aligned}$$

Lemma 1 *Let $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{q}', \tilde{q}'')$ be an optimal solution to (13)–(19) (respectively (13)–(20)). For any $t \in T$ and any $i \in \alpha_t$, set*

$$\begin{aligned}
u_i &= \tilde{u}_i \ , \quad i \in V, & (30)\\
w_e^i &= \tilde{w}_e^i \ , \quad e \in \vec{E}, \ \ i \in \{h(e), t(e)\}, & (31)\\
v_t &= \min_{i \in \alpha_t} (1 - u_i) \ , \quad t \in T, & (32)\\
q'_{it} &= 1 - u_i \ , \quad t \in T, \ \ i \in \alpha_t, & (33)\\
q''_{it} &= 1 - v_t \ , \quad t \in T, \ \ i \in \alpha_t. & (34)
\end{aligned}$$

Then (u, v, w, q', q'') is an optimal solution for (13)–(19) (respectively (13)–(20)).

Proof. Let $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{q}', \tilde{q}'')$ be an optimal solution to (13)–(19). It is easy to see that the problem (13)–(19) with fixed $u = \tilde{u}$ and $w = \tilde{w}$ decomposes into $|T|$ problems of the following form: $(t \in T)$

$$\min \quad A_t v_t + \Delta \sum_{i \in \alpha_t} (q'_{it} + q''_{it}) \ ,$$

subject to (15), (16), (18), and (19). We claim that, for each $t \in T$, the solution (v_t, q', q'') defined by (32)–(34) is feasible and optimal for the corresponding problem. Each of these problems with fixed $v_t = \lambda \in [0, 1]$ converts into a trivial optimization problem whose optimal value as a function of λ is

$$\begin{aligned}
Q(\lambda) &= (\Delta|\alpha_t| - c_t)\lambda + \Delta \sum_{i \in \alpha_t} \max\{2 - \lambda - u_i, 1\}\\
&= \Delta|\alpha_t| - c_t\lambda + \Delta \sum_{i \in \alpha_t} \max\{1 - u_i, \lambda\}.
\end{aligned}$$

The last expression shows that, by the choice of Δ, the function $Q(\lambda)$ is minimized by $\lambda^* = \min_{i \in \alpha_t}(1 - u_i)$. To prove the claim it remains to observe that equalities (33) and (34) give optimal values for q' and q'' provided that u, v and w are defined by (30)–(32). Thus equalities (30)–(34) define an optimal solution to the problem (13)–(19). This solution is clearly binary if the original solution is binary.
□

Let $C = 2\Delta \sum_{t \in T} |\alpha_t| + \sum_{e \in \vec{E}} d_e - \sum_{t \in T} c_t$.

Lemma 2 *Let (u, v, w, q', q'') be an optimal solution to (13)–(19) ((13)–(20)) satisfying (32)–(34). Then $(x, y, z) = \varphi_0(u, v, w, q', q'')$ is feasible for (6)–(9) ((6)–(10)) and*

$$\mathcal{F}(x, y, z) = \mathcal{H}(u, v, w, q', q'') - C \ .$$

Proof. It is clear from (21)–(23) and optimality of (u,v,w,q',q'') that each component of (x,y,z) is nonnegative. Furthermore, the vector (x,y,z) is binary provided that (u,v,w,q',q'') is binary. By (14), for any $e \in \vec{E}$,

$$x_{h(e)} + x_{t(e)} + z_e = u_{h(e)} + w_e^{h(e)} + u_{t(e)} + w_e^{t(e)} - 1 \geq 1 \ .$$

Next, by (32), for any $t \in T$ and any $i \in \alpha_t$,

$$y_t - x_i = 1 - v_t - u_i = 1 - u_i - \min_{k\in\alpha_t}(1-u_k) \geq 0 \ .$$

Finally, by (33) and (34) we have

$$\begin{aligned}
\mathcal{H}(u,v,w,q'.q'') - \mathcal{F}(x,y,z) &= \Delta\sum_{t\in T}|\alpha_t|v_t + \Delta\sum_{i\in V}|\{t : i \in \alpha_t\}|u_i \\
&\quad +\Delta\sum_{t\in T}\sum_{i\in\alpha_t}(2-u_i-v_t) + \sum_{e\in\vec{E}} d_e - \sum_{t\in T} c_t \\
&= 2\Delta\sum_{t\in T}|\alpha_t| + \sum_{e\in\vec{E}} d_e - \sum_{t\in T} c_t \ .
\end{aligned}$$

□

Lemma 3 *If (x,y,z) is an optimal solution to (6)–(9) ((6)–(10)), then $(u,v,w,q',q'') = \psi(x,y,z)$ is feasible for (13)–(19) ((13)–(20)) and*

$$\mathcal{H}(u,v,w,q',q'') = \mathcal{F}(x,y,z) + C \ .$$

Proof. It follows from the definition of ψ and optimality of (x,y,z) that the vector (u,v,w,q',q'') is nonnegative. Furthermore, it is binary if so is (x,y,z). Now by using (7)– (9) and (24)–(29) we check (14)–(18):

$$\begin{aligned}
u_{h(e)} + w_e^{h(e)} &= x_{h(e)} + 1 - x_{h(e)} = 1 \ , \quad e \in \vec{E} \ ; \\
u_{t(e)} + w_e^{t(e)} &= x_{t(e)} + x_{h(e)} + z_e \geq 1 \ , \quad e \in \vec{E} \ ; \\
u_i + q'_{it} &= x_i + 1 - x_i = 1 \ , \quad i \in \alpha_t, \quad t \in T; \\
v_t + q''_{it} &= 1 - y_t + y_t = 1 \ , \quad i \in \alpha_t, \quad t \in T; \\
w_e^{h(e)} + w_e^{t(e)} &= 1 - x_{h(e)} + x_{h(e)} + z_e \geq 1 \ ; \quad e \in \vec{E}; \\
q'_{it} + q''_{it} &= 1 - x_i + y_t \geq 1 \quad i \in \alpha_t, \quad t \in T \ .
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\mathcal{H}(u,v,w,q,q'') - \mathcal{F}(x,y,z) &= \Delta\sum_{t\in T}|\alpha_t| - \sum_{t\in T} c_t - \Delta\sum_{t\in T}|\alpha_t|y_t + \sum_{e\in\vec{E}} d_e \\
&\quad +\Delta\sum_{i\in V}|\{t : i \in \alpha_t\}|x_i + \Delta\sum_{t\in T}\sum_{i\in\alpha_t}(1-x_i+y_t) \\
&= 2\Delta\sum_{t\in T}|\alpha_t| + \sum_{e\in\vec{E}} d_e - \sum_{t\in T} c_t \ .
\end{aligned}$$

□

Lemma 4

(a) *Let* (u, v, w, q', q'') *be an optimal solution to (13)–(19) (respectively (13)–(20)) satisfying (32)–(34). Then* $(x, y, z) = \varphi_0(u, v, w, q', q'')$ *is an optimal solution to (6)–(9) (respectively (6)–(10)).*

(b) *If* (x, y, z) *is an optimal solution of (6)–(9) (respectively (6)–(10)), then* $(u, v, w, q', q'') = \psi(x, y, z)$ *is optimal for (13)–(19) (respectively (13)–(20)).*

Proof. (a) Assume to the contrary that

$$\mathcal{F}(x, y, z) > \mathcal{F}(\tilde{x}, \tilde{y}, \tilde{z})$$

for some $(\tilde{x}, \tilde{y}, \tilde{z})$ optimal for (6)–(9) ((6)–(10)). Then by Lemmas 2 and 3 we must have

$$\mathcal{F}(x, y, z) = \mathcal{H}(u, v, w, q', q'') - C > \mathcal{H}(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{q}', \tilde{q}'') - C = \mathcal{F}(\tilde{x}, \tilde{y}, \tilde{z}) \tag{35}$$

where $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{q}', \tilde{q}'') = \psi(\tilde{x}, \tilde{y}, \tilde{z})$. This contradicts optimality of (u, v, w, q', q'').

(b) Similarly, assume that

$$\mathcal{H}(u, v, w, q', q'') > \mathcal{H}(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{q}', \tilde{q}'')$$

for some $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{q}', \tilde{q}'')$ optimal for (13)–(19) ((13)–(20)). By Lemma 1 we may assume that $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{q}', \tilde{q}'')$ satisfies (32)–(34). Now by Lemmas 2, 3 we have

$$\mathcal{H}(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{q}', \tilde{q}'') = \mathcal{F}(\tilde{x}, \tilde{y}, \tilde{z}) + C < \mathcal{F}(x, y, z) + C = \mathcal{H}(u, v, w, q', q'')$$

where $(\tilde{x}, \tilde{y}, \tilde{z}) = \varphi_0(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{q}', \tilde{q}'')$. This contradicts optimality of (x, y, z).

$\square$

Lemma 5 *Let* (u, v, w, q', q'') *be an optimal solution to (13)–(19) (respectively (13)–(20)). Set*

$$x_i = u_i, \quad i \in V, \tag{36}$$

$$y_t = \max_{i \in \alpha_t} u_i\,, \quad t \in T, \tag{37}$$

$$z_e = w_e^{h(e)} + w_e^{t(e)} - 1\,, \quad e \in \vec{E}. \tag{38}$$

Then (x, y, z) *is an optimal solution of (6)–(9) (respectively (6)–(10)).*

Proof. The statement is an easy consequence of Lemma 1 and the assertion (a) of Lemma 4.

$\square$

Denote by φ the mapping defined by formulas (36)–(38).

3 Proof of Theorem 2

(b1) By the assertion (a1) of Theorem 1 there exists an optimal solution (u, v, w, q', q'') to the linear relaxation (13)–(19) with $u_i, w_e^{h(e)}, w_e^{t(e)} \in \{0, 1/2, 1\}$ for all $i \in V$, $e \in \vec{E}$. By Lemma 5 it follows that the solution $(x, y, z) = \varphi(u, v, w, q', q'')$ is optimal for (6)–(9) and satisfies (12). Furthermore, note that (13)–(19) is an instance of the FVCP on the graph with at most $|V| + |T| + 2|E| + 2|V||T|$ vertices and at most $2|E| + 3|V||T|$ edges. From this, taking into account Remark 1, we conclude that an optimal solution to (13)–(19) satisfying (12) can be found in $O(\max\{|V||T|, |E|\}^{2.5})$ time.

(b2) Let (x, y, z) be an optimal solution to the linear relaxation (6)–(9). Suppose that $x_i = p$ for some $i \in V$ and $p \in \{0, 1\}$. Let $(u, v, w, q', q'') = \psi(x, y, z)$. Then by (b) of Lemma 4 (u, v, w, q', q'') is optimal for (13)–(19) and $u_i = p$. By (a2) of Theorem 1 there exists an optimal solution $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{q}', \tilde{q}'')$ of (13)–(20) such that $\tilde{u}_i = p$. Set $(\tilde{x}, \tilde{y}, \tilde{z}) = \varphi(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{q}', \tilde{q}'')$. By Lemma 5 $(\tilde{x}, \tilde{y}, \tilde{z})$ is optimal for (6)–(10) and $\tilde{x}_i = p$, as desired.

(b3) Suppose that there exist $i \in V$ and $p \in \{0, 1\}$ such that $x_i = p$ for any optimal solution (x, y, z) to (6)–(9). Then by Lemma 5 $u_i = p$ for any optimal solution (u, v, w, q', q'') to (13)–(19). Now by (a3) of Theorem 1 $\tilde{u}_i = p$ for any optimal solution $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{q}', \tilde{q}'')$ to (13)–(20). Finally, by (b) of Lemma 4 $\tilde{x}_i = p$ for any optimal solution $(\tilde{x}, \tilde{y}, \tilde{z})$ to (6)–(10), as desired.

(b4) It follows directly from Lemmas 1–4 that the integrality gap of (6)–(9) is equal to zero if and only if so is the integrality gap of (13)–(20). To obtain the result it remains to use (a4) of Theorem 1 and the fact that (13)–(20) is an instance of the VCP on a graph with at most $|V| + |T| + 2|E| + 2|V||T|$ vertices.

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