Sierpinski's Theorem is Deducible from Euler and Dirichlet

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The famous theorem of Dirichlet says that the sequence

$$an + b, \quad n = 1, 2, \ldots$$

contains infinitely many primes if a and b are relatively prime integers. It is a similar long-standing conjecture that the quadratic sequence

$$n^2 + t, \quad n = 1, 2, \ldots$$

contains infinite number of primes for any positive integer t. The first result in this direction is due to Sierpinski [6] who showed that for any M there exists t' such that the sequence

$$n^2 + t', \quad , n = 1, 2, \ldots$$

contains at least M primes. Recently several new proofs and extensions of Sierpinski's theorem have been appeared [4][3][1].

In this note we show that a slightly stronger result can be easily derived from the following well-known number theory facts:

Theorem 1 (Euler) ([5, Theorem 251]) *Every prime of the form $4n+1$ is representable as a sum of two squares. This representation is unique up to the order of the summands.*

Theorem 2 (Dirichlet) ([2, p. 34]) *The series*

$$\sum_{\text{prime } p \equiv b \pmod{a}} 1/p$$

where a, b are relatively prime integers, diverges.

Denote by P the set of primes of the form $4n+1$. Note that by Theorem 1 the set of primes in $\{(2k+1)^2+4l^2 \mid k,l \in \mathbf{Z}_+\}$ coincides with P. For any $k \in \mathbf{Z}_+$, denote by P_k the set of primes in $\{4l^2+4k(k+1)+1 \mid l \in \mathbf{Z}_+\}$. Now Theorem 1 can be reformulated in the following way:

$$\bigcup_{k\in\mathbf{Z}_+} P_k = P, \tag{1}$$

$$P_{k'} \cap P_{k''} = \emptyset \quad \text{for all } k', k'' \in \mathbf{Z}_+ \text{ such that } k' \neq k''. \tag{2}$$

Theorem 3 *For any $k_0, M \geq 0$ and any $\delta \in [0,1[$ there exists a positive integer $k^* \geq k_0$ such that*

$$|P_{k^*}| > M(2k^*+1)^\delta.$$

(To obtain Sierpinski's theorem, take $k_0 = \delta = 0$.)

Proof. By (1) and (2) we may formally write

$$\sum_{p\in P} 1/p = \sum_{k\in\mathbf{Z}_+} \sum_{p\in P_k} 1/p\ . \tag{3}$$

Note first that

$$\sum_{p\in P_k} 1/p < \sum_{l\in\mathbf{Z}_+} \frac{1}{4l^2+1} < +\infty \quad \forall k \in \mathbf{Z}_+\ . \tag{4}$$

Now assume to the contrary that, for some $M, k_0 \geq 0$ and some $\delta \in [0,1[$,

$$|P_k| \leq M(2k+1)^\delta \quad \forall k \in \mathbf{Z}_+ \cap [k_0, +\infty[\ .$$

This implies that, for any $k \in \mathbf{Z}_+ \cap [k_0, +\infty[$,

$$\sum_{p\in P_k} 1/p \leq \frac{|P_k|}{4k(k+1)+1} \leq \frac{M}{(2k+1)^{2-\delta}} \leq \frac{M}{(k+1)^{2-\delta}}\ . \tag{5}$$

It follows from (4) and (5) that the series in the right hand side of (3) converges and thereby so does the series in the left hand size. But this contradicts the statement of Theorem 2. $\square$

The following generalization is in fact an easy consequence of Theorem 3.

Corollary 1 *For any $k_0, M, r, s \geq 0$ and any $\delta \in [0,1[$ there exists a positive integer $k^* \geq k_0$ such that*

$$|P_{k^*}| > M(rk^*+s)^\delta. \quad \square$$

References

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