An Approximation Scheme for the Uncapacitated Facility Location Problem on Planar Graphs

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Abstract. We present an approximation scheme for the planar graph case of the Uncapacitated Facility Location Problem. The scheme is based on an adaptation of the general approach due to Baker and becomes a polynomial-time approximation scheme (PTAS) when all ratios of the transportation cost along an edge to the cost of opening a facility at an endpoint of this edge are bounded from below by a fixed positive constant.

Keywords: facility location, polynomial-time algorithm, approximation scheme, planar graph.

Introduction

We consider a special, network case of the uncapacitated facility location problem (the network UFLP). In a formal setting an instance of the network UFLP consists of an undirected connected graph $G = (V, E)$, nonnegative vertex costs $c(v)$, nonnegative vertex demands $\varphi(v)$, and nonnegative edge lengths $l(e)$. It is required to find a nonempty subset $X \subseteq V$ minimizing

$$F(X) = \sum_{v \in X} c(v) + \sum_{w \in V} \varphi(w) \min_{u \in X} d(u, w),$$

where $d(u, w)$ is the length of a shortest path between vertices u and w with respect to the edge lengths $l(e)$.

A common interpretation of the network UFLP is as follows. The graph G represents a transportation network with consumers of some product placed on its vertices; $c(v)$, a fixed cost for opening (locating) a facility at vertex v; $d(v, w)$, a cost for transporting a unit of the product from vertex v to vertex w along a shortest path; $\varphi(v)$, a demand for the product at vertex v. It is required, provided all the demands must be satisfied, to open (or locate) facilities on some vertices of G so as to minimize the total costs for opening facilities and transporting the product to the consumers.

The network UFLP is a special case of the general UFLP in which the transportation costs matrix can be arbitrary. The general UFLP is one of the core models in operations research and has been extensively treated in the literature for the last decades (see the chapter by Cornuejols, Nemhauser, and Wolsey in [8]). The network UFLP is known to be NP-hard on general graphs [10] and this remains true even for grids [2] which are planar graphs of very special type. Moreover, the network UFLP on general graphs can be easily shown to be equivalent to the metric UFLP (in this paper we prefer to use the term 'network' to underscore the essential role of the underlying graph in our treatment). Constant-factor approximation algorithms for solving the latter have been constructed quite recently — see Shmoys, Tardos, and Aardal [11], Guha and Khuller

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[9], and Chudak and Shmoys [7]. At the time of this writing the best approximation ratio of 1.736 is provided by the algorithm of Chudak and Shmoys [7].

In this paper we describe a family of $\left(1+\frac{1}{k\sigma}\right)$-approximation algorithms for solving the network UFLP on planar graphs, where k is a positive integer parameter of the family and

$$\sigma = \min_{v\in V} \frac{\varphi(v)\min_{e\in E(v)} l(e)}{c(v)}$$

($E(v)$ denotes the set of edges incident with v). We show that, for any particular k, the corresponding algorithm of the family runs in polynomial time, and so this family turns into a PTAS on any subset of instances satisfying $\sigma \geq const > 0$.

1. Basic Definitions and Lemmas

A graph is called *planar* if it can be embedded in the plane in such a way that no two edges cross except for their endpoints. Such an embedding is called a *plane embedding.* A graph is called 1-*outerplanar* or simply *outerplanar* if it admits a plane embedding in which the outer face contains all vertices. A graph G is called k-*outerplanar* if $G - X$ is an $k-1$-outerplanar graph where X is the set of vertices belonging to the outer face of G.

The following algorithmic facts provide an opportunity for applying Baker's method to UFLP.

A *tree decomposition* of a graph $G = (V, E)$ is a pair $(\mathcal{X}, T)$ where $T = (I, F)$ is a tree and $\mathcal{X}$ is a family of subsets of V, one for each vertex of T, such that

(i) $\cup_{i\in I} X_i = V$;

(ii) for any edge $vw \in E$, there exists $i \in I$ with $v \in X_i$ and $w \in X_i$;

(iii) for any $i, j, k \in I$, if j lies on the path of T between i and k then $X_i \cap X_k \subseteq X_j$.

The width of a tree decomposition $(\mathcal{X}, T)$ is $\max_{i\in I} |X_i| - 1$. The treewidth of a graph G is the minimum width over all tree decompositions of G.

LEMMA 1 ([4; 5]). *Let G be a k-outerplanar graph. Then*

(i) G has treewidth at most $3k-1$;

(ii) The tree decomposition of G of width $3k-1$ can be found in linear time.

LEMMA 2 ([1]). *UFLP on graphs with treewidth at most r can be solved in $O(n^{r+2})$ time.*

Together with Lemma 1 this yields

LEMMA 3. *UFLP on k-outerplanar graphs can be solved in $O(n^{3k+1})$ time.* □

2. An Adaptation of Baker's Scheme for UFLP

A general approach of designing approximation schemes for some optimization problems on planar graphs was developed by Baker [3]. It is based on the observation that every planar graph can be split into slices each of which is an outerplanar graph and that some optimization problems can be solved in polynomial time if the number of these slices does not exceed a fixed constant k (i.e., the graph is k-outerplanar). We further show how to modify Baker's scheme to make it applicable to UFLP.

Fix some plane embedding of G and construct a sequence $W_1, W_2, \dots, W_s$ of disjoint vertex subsets of G in the following way. Set W_1 to be the set of vertices lying on the boundary of the outer face of G and for the remaining $i = 2, \dots s$, set W_i to be the set of vertices lying on the boundary of the outer face of $G - W_1 - \dots - W_{i-1}$. Such a sequence can be found in linear time by a breadth search procedure. Note that each W_i induces an outerplanar subgraph in G and these subgraphs are linked in a chain-like fashion. Namely, the vertices in W_1 have neighbors only in W_1 and W_2, the vertices in W_2 have neighbors only in W_1, W_2 and W_3, and so on. We will refer to the sets in the sequence $W_1, W_2, \dots, W_s$ as *outerplanar layers of* G . Note that the union of the sets in any segment of the sequence $W_1, W_2, \dots, W_s$, consisting of r sets, induces an r-outerplanar graph in G. Now fix a positive integer $k \geq 2$ and for $i = 0, \dots, k-1$, set

$$X_i = \bigcup_{t \equiv i \pmod{k}} W_t. \tag{1}$$

Then $\{X_1, X_2, \dots, X_k\}$ is a partition of V. Moreover, by definition $G - X_i$ is a $(k-1)$-outerplanar graph for each i.

Consider now an instance of UFLP on a planar graph $G = (V, E)$. Fix a planar embedding of G and consider a sequence of outerplanar layers $\{W_1, \dots, W_s\}$ of G. For any k, $2 \leq k \leq s$, let $\{X_1, X_2, \dots, X_k\}$ be the partition of V defined by (1). Fix k and i, $1 \leq i \leq k$, and consider the instance of UFLP with the additional constraint $X \supseteq X_i$. Our purpose further is to show that this constrained UFLP can be solved in polynomial time. Indeed, set

$$\begin{aligned}
U_0 &= W_1 \cup W_2 \cup \dots \cup W_i \\
U_1 &= W_i \cup W_{i+1} \cup \dots W_{i+k} \\
&\dots \\
U_{l+1} &= W_{i+lk} \cup W_{i+lk+1} \cup \dots \cup W_s
\end{aligned}$$

where $l = \lfloor (s-1)/k \rfloor$. Note that the defined subsets are segments of the sequence of outerplanar layers of G, and, moreover, U_t and U_{t+1} have in common exactly one layer which is a subset of X_i. We claim that our problem decomposes into $l+2$ problems on the induced subgraphs $G[U_0], G[U_1], \dots, G[U_{l+1}]$ with the same edge lengths l, the same demands φ, and new vertex costs c' defined by $c'(v) = c(v)$ if $v \notin X_i$ and $c'(v) = 0$ otherwise. This is indeed the case, since the constrained problem has an optimal solution in which all consumers on the vertices in U_t, for each t, are served only by facilities lying on some vertices in the same set U_t (recall that all facilities on the vertices in X_i are necessarily open). By construction, the subgraphs $G[U_t]$ are $(k+1)$-outerplanar. Thus, applying Lemma 3, we arrive at the desired conclusion.

For each i, let Z_i be an optimal solution of UFLP with the additional constraint $Z_i \supseteq X_i$. Denote by Z^* an optimal solution to the original instance of UFLP. Then $F(Z_i) \leq F(Z^* \cup X_i)$ for all i. That is, for each i,

$$F(Z_i) \;\leq\; c(X_i) + c(Z^* \setminus X_i) + \sum_{v \in V} \varphi(v) \min_{u \in Z^* \cup X_i} d(u,v).$$

Adding these inequalities together over all i we obtain

$$\begin{aligned}
\sum_{i=1}^{k} F(Z_i) \;&\leq\; c(V) + \sum_{i=1}^{k} c(Z^* \setminus X_i) + \sum_{i=1}^{k} \sum_{v \in V} \varphi(v) \min_{u \in Z^* \cup X_i} d(u,v) \\
&\leq\; c(V) + (k-1)c(Z^*) + \sum_{i=1}^{k} \sum_{v \in V} \varphi(v) \min_{u \in Z^*} d(u,v) \\
&\leq\; c(V) + kc(Z^*) + k \sum_{v \in V} \varphi(v) \min_{u \in Z^*} d(u,v) \\
&=\; c(V) + kF(Z^*). \qquad (2)
\end{aligned}$$

Our algorithm calls the algorithm stated in Lemma 3 to find Z_i for each i, and outputs a cheapest of these sets. Let Z' denote that solution. Then by (2)

$$F(Z') \leq 1/k \sum_{i=1}^{k} F(Z_i) \leq c(V)/k + F(Z^*). \qquad (3)$$

We next evaluate a lower bound on the ratio $F(Z^*)/c(V)$. If $u \neq v$ then $d(u,v) \geq \min_{e \in E(v)} l(e)$. Therefore,

$$\begin{aligned}
F(Z^*) \;&=\; \sum_{v \in Z^*} c(v) + \sum_{v \in V \setminus Z^*} \varphi(v) \min_{u \in Z^*} d(u,v) \\
&\geq\; \sum_{v \in V} \min\{c(v), \varphi(v) \min_{e \in E(v)} l(e)\}.
\end{aligned}$$

Set

$$\sigma = \min_{v \in V} \frac{\varphi(v) \min_{e \in E(v)} l(e)}{c(v)}.$$

If $\sigma > 1$, then the cheapest solution is to open a facility at each vertex, i. e., in this case V is the unique optimal solution to the instance . Clearly, we may exclude this trivial case by preprocessing and assume further that $\sigma \leq 1$. Then

$$\begin{aligned}
F(Z^*) \;&\geq\; \sum_{v \in V} \min\{c(v), \sigma c(v)\} \\
&\geq\; \sigma \sum_{v \in V} c(v) \\
&=\; \sigma c(V).
\end{aligned}$$

It follows that $c(V) \leq F(Z^*)/\sigma$ and by (3)

$$F(Z') \leq \left(1 + \frac{1}{k\sigma}\right) F(Z^*), \qquad (4)$$

that is the approximation ratio of our algorithm is bounded by $\left(1+\frac{1}{k\sigma}\right)$ as was claimed above.

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