Complexity of the Network Median Problem on Grids

A. A. Ageev*

Abstract

The n-grid is defined to be the adjacency graph of the $n \times n$ integer lattice. A graph is called a planar grid if it is isomorphic to a subgraph of the n-grid. The paper shows that the network median problem (an analog of the well-known k-median problem) remains NP-hard when restricted to the set of n-grids.

1 Introduction

Let $P(\Gamma)$ be an optimization or decision problem whose instance includes a graph taken from a prescribed set of undirected graphs Γ. Choosing a subset $\Gamma' \subseteq \Gamma$ and considering the problem $P(\Gamma)$ on this subset we obtain a subproblem $P(\Gamma')$. Even though $P(\Gamma)$ is NP-hard (NP-complete), the subproblem $P(\Gamma')$ may turn out to be polynomially solvable for some Γ'. The question of complete characterization of such sets of graphs is of greatest importance both for theoretical developments and practical applications. In its general setting this question appears to be very difficult and remains unresolved for any concrete problem. However, the situation may become much easier if we impose some restrictions on the families of sets. It is a natural restriction for a variety of applications to consider only minor closed sets of graphs.

A *minor* of a graph G is a graph which is obtained from a subgraph of G by contracting some edges.

A set of graphs Γ is *minor-closed* if for every $G \in \Gamma$ every minor of G is also in Γ.

*Supported in part by the Russian Foundation for Fundamental Research under Grant No 93-011-1486

Many well-known sets of graphs (forests, planar, outerplanar, series-parallel graphs etc) are minor-closed.

There is a theorem due to N. Robertson and P. D. Seymour [11] establishing a connection between the notion of graph minor and a graph parameter termed treewidth.

Let $G = (V, E)$ be a graph. A *tree-decomposition* χ of G is a pair $(T, \{X_t : t \in V(T)\})$ where T is a tree, $\{X_t : t \in V(T)\}$ is family of subsets of V such that

(i) $\bigcup(X_t : t \in V(T)) = V$

(ii) for each $e \in E$ there exists $t \in V(T)$ such that e has both ends in X_t;

(iii) for $l, l', l'' \in L$, if l' lies on the path of T between l and l'' then $X_t \cap X_{l''} \subseteq X_{l'}$.

The *width* of the tree-decomposition is $\max(|X_t| - 1 : t \in V(T))$. The *tree-width of G* to be denoted by $tw(G)$ is the minimum w such that G has a tree-decomposition of width $\leq w$.

The following theorem is indeed a reformulation of the main result in [11] (see also [2]).

Theorem 1. *Let Γ be a minor-closed set of graphs which does not include all planar graphs. Then there exists an integer q such that for any graph $G \in \Gamma$ $tw(G)$ does not exceed q.*

Corollary 1. *Suppose that a problem P is NP-hard on planar graphs. Suppose also that for every fixed q the problem P is polynomially solvable on the set of graphs having treewidth $\leq q$. Let Γ be a minor-closed set of graphs. Then*

(a) *if Γ does not contain all planar graphs, then P polynomially solvable;*

(b) *if Γ contains all planar graphs then P is NP-hard.*

Many problems being NP-hard on general graphs turns out to be polynomially solvable on graph with bounded treewidth [3], [4], [6], [7]. Some of them are NP-hard (NP-complete) on planar graphs (VERTEX COVERING, DOMINATING SET, Steiner problem etc, see [1]). By Corollary 1 for the problems of this kind an easy verifiable criterion of polynomial-time solvability holds (namely, assertions (a),(b) of Corollary

1). Originally this observations were applied in [2] to the network median problem (NMP).

The NMP is, given a connected undirected graph G whose vertices have nonnegative weights $c(v)$ and $\varphi(v)$, $v \in V(G)$ and edges nonnegative lengths $l(e)$, $e \in E(G)$,, to find nonempty vertex subset $X \subseteq V(G)$ minimizing

$$\Phi(X) = \sum_{v \in X} c(v) + \sum_{w \in V} \varphi(w) \min_{v \in X} d_{vw}, \tag{1}$$

where d_{vw} is the shortest distance matrix of G with respect to the length function l.

Theorem 2 ([2]). *Let Γ be a minor-closed set of graphs. Then*

(a) *if Γ does not contain all planar graphs then the NMP is polynomially solvable;*

(b) *if Γ contains all planar graphs then the NMP is NP-hard.*

The restriction for a set of graphs to be minor-closed seems to be natural for the NMP: it is easy to see that prescribing a sufficiently large length to an edge is equivalent to its deletion and prescribing zero length to its contraction. Nevertheless, not all sets of graphs arising in applications are minor-closed. Regular graphs and grids are few examples.

2 The main result

Let Γ be a set of graphs. Define *minor closure* $\bar{\Gamma}$ to be the intersection of all minor-closed sets containing Γ. Let P be a problem on graphs and $P(\Gamma)$ be the restriction of P to Γ. Now if $P(\bar{\Gamma})$ is polynomially solvable then so is $P(\Gamma)$. However, nothing can be said about the complexity of $P(\Gamma)$ if $P(\bar{\Gamma})$ is NP-hard (NP-complete). This holds, in particular, in case P satisfies the conditions of Corollary 1 and Γ includes all planar graphs.

(Planar) n-grid is the undirected graph G with vertex set $\{v_{ij} : 1 \leq i, j \leq n\}$ in which v_{ij} $v_{i'j'}$ are adjacent if and only if $|i - i'| + |j - j'| = 1$.

In other words, n-grid is the adjacency graph of $n \times n$ integer lattice.

Call a graph G a *partial grid* if it is isomorphic to a subgraph of a n-grid for some n.

The minor closure of the set of partial grids is known to be equal to the set of planar graphs [5]. Hence, restricting the NMP to partial grids we get the situation described above. Unfortunately, this case of the problem remains NP-hard.

Theorem 3. *The NMP is NP-hard on the set of n-grids.*

Corollary 2. *The NMP is NP-hard on the set of partial grids.*

Theorem 3 will be proved in Section 4. The next section contains three lemmas on which this proof is based.

3 Some lemmas

The graphs considered are undirected and may not have loops. If G is a graph then $V(G)$ denotes the set of vertices of G, and $E(G)$ the set of edges.

The vertex subset X of G is called *dominating* if every vertex of G is either in X or adjacent to some vertex in X.

The simple path (cycle) with vertex set $\{v_1, \ldots, v_p\}$ will be denoted by $(v_1, \ldots, v_p)$ $((v_1, \ldots, v_p, v_1))$.

The NMP is well-known to be NP-hard on general graphs [9][1]. It can be shown that the NMP remains NP-hard on planar graphs [2]. In the proof of Theorem 3 we shall need the following refinement of this result.

Lemma 1. *The NMP is NP-hard on the set of planar cubic graphs.*

Proof. Using the fact that the minimum weight dominating set problem (DSP) is NP-hard on planar cubic graphs [10] present a polynomial time reduction of the latter problem to the NMP on planar cubic graphs.

Suppose that a planar cubic graph G on n vertices and a function $w : V \to \mathbf{R}_+$ form an instance of the DSP.

Consider the following instance of the NMP : a graph G, $\varphi(v) = 1$, $c(v) = \Delta + w(v)$, $l(e) = \Delta$, $v \in V(G)$, $e \in E(G)$, where Δ is a number exceeding $\sum_{v \in V} w(v)$. Let X be an optimal solution of this instance. Let us show that X is a dominating set with the minimum weight (i.e.

$$\sum_{v \in X} w(v) = \min\{\sum_{v \in Y} w(v) : \ Y \text{ is a dominating set of } G\}).$$

Indeed, observe that $\Phi(Y) \geq n\Delta$ for every nonempty vertex subset $Y \subseteq V$. Furthermore, clearly,

$$\Phi(Y) < (n+1)\Delta \tag{2}$$

if and only if Y is a dominating set. By optimality of X it follows that X is a dominating set. By the choice of Δ the inequality (2) is equivalent to

$$\Phi(Y) = n\Delta + \sum_{v \in Y} w(v),$$

from which we conclude that X has minimum weight. □

Remark 1. In the proof above as is easily seen we does not use the fact that G is cubic and planar. This means that if the DSP is shown to be NP-hard on a set of graphs then the NMP must be NP-hard on the same set of graphs.

A graph H is called a *subdivision* of a graph G if H is obtained from G after replacing some edges by disjoint simple paths. We say that a graph H *topologically contains* a graph G if H has a subgraph isomorphic to a subdivision of G. Clearly, if H topologically contains G then G is a minor of H.

We call a pair of injections (φ, ψ) a *drawing* of a graph H if $\varphi : V(H) \to \mathbf{Z^2}$ maps vertices of H into integral points of the plane, nd ψ transforms every edge $e = v_1v_2 \in E(H)$ into a simple curve having the endpoints $\varphi(v_1)$ and $\varphi(v_2)$ and consisting of horizontal and vertical segments connecting neighboring integral points of the plane.

Lemma 2. *Given a cubic planar graph G, one can find in polynomial time a drawing (φ, ψ) such that*

(a) *the distances between the images of the vertices of G (with respect to l_∞) under the mapping φ are bounded by $O(|V(G)|)$;*

(b) *the curves which are the images of the edges of G under the mapping ψ are disjoint in the interior points.*

Remark 2. Clearly, every drawing of the graph G, satisfying the conditions (a), (b) defines a subdivision G^* of a graph G such that G^* is a subgraph of a $O(|V(G)|)$-grid H. Moreover, it can be easily seen that using this drawing one can transform G^* into H in linear time.

Proof of Lemma 2. Let G be a cubic planar graph. It is easy to see that in linear time one can find a connected cubic graph G' having no bridge and such that G is topologically contained in G', $|V(G')| = |V(G)| + 2r + 4(q-1)$, where r and q are the number of bridges and of components in G respectively. Hence without loss of generality we may assume that G is connected and has no bridge. It follows that in every plane embedding of G the boundary of every face is a cycle.

Suppose that we have a plane embedding of G (recall that it can be found in linear time [8]). Let $\{C_1, \ldots, C_p\}$ be the set of cycles bounding the faces of G in this embedding and numbered in such a way that

$$\bigcup_{k=1}^{i} V(C_k) \cap V(C_{i+1}) \neq \emptyset \text{ for any } i \in \{1, \ldots, p-2\}. \tag{3}$$

For any $i = 1, \ldots, p-1$, denote by H_i a subgraph of G with vertex set $V(H_i) = \bigcup_{j=1}^{i} V(C_j)$ and edge set $E(H_i) = \bigcup_{j=1}^{i} E(C_j)$. Clearly, H_i are plane graphs, $H_1 = C_1$ and $H_{p-1} = G$. Denote by B_i the cycle bounding the infinite face of H_i. Suppose that the cycles are numerated in such a way that in addition to (3) the following holds:

for any $i = 1, \ldots, p-2$ the intersection of the cycles B_i C_{i+1} is a path (4)

It is easy to see that a numeration satisfying (3), (4) does exist and can be found in polynomial (even in linear) time. Hence the assumption stated above does not restrict generality.

Now we present a informal description of an algorithm for finding a drawing (φ, ψ) satisfying (a), (b).

The algorithm consists of $p-1$ steps. The i's step consists of constructing the image (φ_i, ψ_i) of the graph $H_i (\varphi, \psi) = (\varphi_{p-1}, \psi_{p-1})$.

Step 1. Let $C_1 = (x_1, \ldots, x_k, x_1)$. Set $\varphi_1(x_t) = (t, 1)$. The points $(t, 1)$ and $(t+1, 1)$ are linked by the horizontal segments for every $t = 1, \ldots, k-1$. The points $(1,1)$ and $(1,k)$ are linked by the broken line consisting of the following five segments: $\Big((1,1),(0,1)\Big)$, $\Big((0,1),(0,0)\Big)$, $\Big((0,0),(k+1,0)\Big)$, $\Big((k+1,0),(k+1,1)\Big)$, $\Big((k+1,1),(k,1)\Big)$.

Step i ($i = 2, \ldots, p-1$). Before the step we have the desired image of the graph H_{i-1} $(\varphi_{i-1}, \psi_{i-1})$ satisfying the following property: the outer cycle ${}_{i-1}$ of the graph H_{i-1} maps into the rectangle with the vertices $(0,0)$, $(0,\mu)$, $(\lambda+1,\mu)$, $(\lambda+1,0)$ while the vertices of the cycle

$_{i-1}$ map into some integer points of the upper base of this rectangle: $(k_1,\mu),(k_2,\mu),\dots,(k_r,\mu)$, $1 \le k_1 < k_2 < \dots < k_r \le \lambda$. Let

$$C_i = (u_1,\dots,u_\alpha,w_1,\dots,w_\beta,u_1),$$

where $\alpha \ge 2$, $\beta \ge 0$, $\{w_1,\dots,w_\beta\} = V(C_i) \setminus V(B_{i-1})$.

It follows from the conditions (3), (4) that the following two cases are possible.

CASE 1: *the image of the set* $\{u_1,\dots,u_\alpha\}$ *under the mapping* φ_{i-1} *is the set of points* $\{(k_j,\mu) : s' \le j \le s''\}$. The image $(\varphi_{i-1},\psi_{i-1})$ is transformed into the image (φ_i,ψ_i) in the two stage:

$$(\varphi_{i-1},\psi_{i-1}) \to (\varphi_{i,1},\psi_{i,1}) \to (\varphi_i,\psi_i),$$

where $(\varphi_{i,1},\psi_{i,1})$ is an image of the graph H_i.

The restriction of the image $(\varphi_{i,1},\psi_{i,1})$ to the graph H_{i-1} is obtained from the image $(\varphi_{i-1},\psi_{i-1})$ by the horizontal expansion of the plane

$$\sigma\big((x,y)\big) := \begin{cases} (x,y)\ , & \text{if } x \le s'; \\ (x+\beta,y)\ , & \text{otherwise.} \end{cases}$$

For the graph H_i set

$$\varphi_{i,1}(v_j) = (k_{s'}+j,\mu+1),\ \ j = 1,\dots,\beta.$$

Finally, we complete constructing the image $\psi_{i,1}$ in a way to satisfy the requirement that the image of the path $(u_1,w_\beta,\dots,w_1,u_\alpha)$ is the broken line

$$\big((k_{s'},\mu),(k_{s'},\mu+1),(k_{s''}+\beta,\mu+1),(k_{s''}+\beta,\mu)\big).$$

The second stage ("straightening") starts with setting

$$\varphi_i(v) := \begin{cases} \varphi_{i,1}(v) + (0,1)\ , & v \in V(B_{i-1}) \setminus \{u_2,\dots,u_{\alpha-1}\}; \\ \varphi_{i,1}(v)\ , & \text{otherwise.} \end{cases}$$

where $(0,1)$ is treating as vector. Next, we set $\psi_i(e) = \psi_{i,1}(e)$ for all $e \in E(H_i)$ such that e is incident to no vertex of the cycle B_i. For the edges of H_i which are incident to at least one vertex of the cycle $_i$, the mapping ψ_i is defined as follows: the cycle B_i maps into the rectangle $\big((0,0),(0,\mu+1),(\lambda+\beta+1,\mu+1),(\lambda+\beta+1,0)\big)$, and the broken lines corresponding to the other edges lengthen (if necessary) upwards by a unit segment.

Case 2: *the image of the set* $\{u_1, \ldots, u_\alpha\}$ *under the image* φ_{i-1} *is the set* $\{(k_j, \mu) : j \leq s', j \geq s''\}$. Denote by $(\varphi_{i-1,1}, \psi_{i-1.1})$ the image of the graph H_{i-1} which is obtained from the image $(\varphi_{i-1}, \psi_{i-1})$ by the parallel shift of the plane by a vector $(1+\beta, 1)$. The image (φ_i, ψ_i) is obtained from the image $(\varphi_{i-1,1}, \psi_{i-1.1})$ in the two stages:

$$(\varphi_{i-1,1}, \psi_{i-1,1}) \to (\varphi_{i,1}, \psi_{i,1}) \to (\varphi_i, \psi_i),$$

where $(\varphi_{i,1}, \psi_{i,1})$ is an image of the graph H_i. The restriction of the image $(\varphi_{i,1}, \psi_{i,1})$ to the graph H_{i-1} is defined in such a way to provide coincidence with the image $(\varphi_{i-1,1}, \psi_{i-1,1})$. Next, we set

$$\varphi(w_j) = (j, \mu+2), \quad j = 1, \ldots, \beta.$$

Now we finish defining the image $\psi_{i,1}$ by the condition that the path $(u_1, w_1, \ldots, w_\beta, u_\alpha)$ transforms into the broken line $\big((k_{s'} + \beta + 1, \mu + 1), (k_{s'} + \beta + 1, \mu + 2), (0, \mu + 2,), (0, 0), (\lambda + \beta + 2, 0), (\lambda + \beta + 2, \mu + 2), (k_{s''} + \beta + 1, \mu + 2), (k_{s''} + \beta + 1, \mu + 1)\big)$. The second stage is analogous to that of the case 1. First, set

$$\varphi_i(v) := \begin{cases} \varphi_{i,1}(v) + (0,1)\ , & v \in V(B_{i-1}) \setminus \{u_2, \ldots, u_{\alpha-1}\}; \\ \varphi_{i,1}(v)\ , & \text{otherwise.} \end{cases}$$

Then set $\psi_i(e) = \psi_{i,1}(e)$ for every edge $e \in E(H_i)$ such that e is incident to no vertex of the cycle B_i. Now we finish defining the image ψ_i by the condition that the cycle B_i transforms into the rectangle with the vertices $(0,0), (\lambda+\beta+2, 0), (\lambda+\beta+2, \mu+2), (0, \mu+2)$. The broken lines, corresponding to the edges of H_i which are incident to some vertices of the cycle B_i but not contained in it, lengthen upwards by a unit segment.

It is easy to see that at the i-s step the length of the rectangle corresponding to the outer face of the graph H_i, increases at most by $\beta + 2$ and the height at most by 2. Since by Euler equation for any cubic plane graph we have $p = 1/2|V(G)|+2$ it follows that the length and the height of the terminal rectangle do not exceed $2(p-1)+|V(G)| = 2(|V(G)|+1)$. Thus the image (φ_i, ψ_i) satisfies the condition (a). This also implies that the running time of the algorithm is polynomially bounded (in fact as is easily seen the algorithm can be implemented in linear time). On the other hand it follows directly from the description of the algorithm that every image $(\varphi_i, \psi_i)(i = 1, \ldots, p-1)$, including $(\varphi, \psi) = (\varphi_{p-1}, \psi_{p-1})$ satisfies the condition (b).

Thus, the proof of Lemma 2 is complete. $\square$

Bergmann [5] showed that every cubic planar graph G is a minor of a r-grid for a sufficiently large r. In his proof r has the order of $2^{|V(G)|}$. It is easy to see that our proof of Lemma 2 contains an implicit proof of a stronger result.

Corollary 3. *Every planar cubic graph G is topologically contained in a $2(|V(G)| + 2r + 4(q-1) + 1)$-grid where r and q are the numbers of bridges and components of G relatively.*

Let G and H be connected graphs such that H topologically contains G. For any edge $e \in E(G)$, denote by $F(e)$ the set of edges of the path in H which substitutes e. Let W denote the set of inner vertices in these paths.

Suppose now that we are given an instance of the NMP on G with the functions $c, \varphi : V(G) \to \mathbf{R}_+$, $l : E(G) \to \mathbf{R}_+$. Consider the instance of the NMP on H with the weight functions c_H, $\varphi_H : V(H) \to \mathbf{R}_+$, $l_H : E(H) \to \mathbf{R}_+$ defined by the following equations:

$$\varphi_H(v) = \begin{cases} \varphi(v)\,, & \text{if } v \in V(G), \\ 0\,, & \text{otherwise;} \end{cases} \tag{5}$$

$$c_H(v) = \begin{cases} c(v)\,, & \text{if } v \in V(G), \\ \Phi\,, & \text{if } v \in W, \\ 0\,, & \text{otherwise;} \end{cases} \tag{6}$$

$$l_H(f) = \begin{cases} l(e)/|F(e)|\,, & \text{if } f \in F(e),\, e \in E(G), \\ \Phi\,, & \text{otherwise;} \end{cases} \tag{7}$$

where Φ is a number exceeding $\sum_{e \in E(G)} l(e)$.

The following lemma is easily implied by the construction.

Lemma 3. *If $Y \subseteq V(H)$ is an optimal solution of the above defined instance of the NMP on H then $Y \cap V(G)$ is an optimal solution of the original instance of the NMP on G.* □

4 Proof of Theorem 3

By Lemma 1 the NMP on cubic planar graphs is NP-hard. Let us show that the NMP on cubic planar graphs can be transformed in polynomial time to the NMP on grids. Consider the instance of the NMP (G, c, φ, l) where G is a cubic planar graph. By Lemma 2 (see Remark 2) one can find in polynomial (even linear) time a graph G^* which is a subgraph of a $O(|V(G)|)$-grid H and a subdivision of G. Applying formulas (6)–(7) transform the instance (G, c, φ, l) to the instance (H, c_H, φ_H, l_H). By Lemma 3 this transformation is exactly that we need. $\square$

References

[1] M. R. Garey and D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness. Freeman, San Francisco, 1979.

[2] Ageev A. A. A criterion of polynomial-time solvability for the network location problem // Integer Programming and Optimization. Campus Printing, Carnegie Mellon University.—1992.— P. 237–245.

[3] Arnborg S., Lagergen J., Seese D. Easy problems for tree-decomposable graphs // J. Algorithms —1991.—V. 12. —P. 308–340.

[4] Arnborg S., Proskurowski A. Linear time algorithms for NP-hard problems restricted to partial q-trees // Discr. Appl. Math. —1989.—V. 23—P. 11–24.

[5] Bergmann H. Ein Planaritätskriterium für endliche Graphen // Elem. Math.—1982.—V. 37.—P. 49–51.

[6] Bern W., Lawler E. L.,Wong L. Linear time computation of optimal subgraphs of decomposable graphs // J. Algorithms —1987.—V. 8.—P. 216–235.

[7] Bodlaender H. L. Dynamic programming on graphs with bounded treewidth // Lecture Notes in Comput. Sci. 317, Springer-Verlag, Berlin.—1988.—P. 105–118.

[8] Hopcroft J. E., Tarjan R. E. Efficient planarity testing // J. Assoc. Comput. Mach.—1974.—V. 21.—P. 549–568.

[9] Kariv O., Hakimi S. L. An algorithmic approach to network location problems. Part 2: The p-medians // SIAM J. Appl. Math. —1979.— V. 37.— P. 539–560.

[10] Kikuno T., Yoshida N., Kakuda Y. The NP-completeness of the dominating set problem in cubic planar graphs // Trans. IEEE Japan —1980.—V. E64.—P. 434-444.

[11] Robertson N., Seymour P. D. Graph minors. V. Excluding a planar graph // J. Comb. Theory—1986.—V. 41B.—P. 92–114.