A Triangle-Free Circle Graph with Chromatic Number 5

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Abstract

It follows from the results of Karapetyan (1984), Gyárfás and Lehel (1985), and Kostochka (1988) that

$$4 \leq \chi^* \leq 5$$

where $\chi^* = \max\{\chi(G) : G \text{ is a triangle-free circle graph}\}$. We show that $\chi^* \geq 5$ and thus $\chi^* = 5$. This disproves the conjecture of Karapetyan that $\chi^* = 4$ and answers negatively a question of Gyárfás and Lehel.

A graph is called a *circle* graph if it is isomorphic to the intersection graph of chords of a circle. Since any complete graph is obviously a circle graph, the chromatic number of an arbitrary circle graph can be arbitrarily large. Gyárfás [1] showed that the chromatic number of a circle graph G without cliques of size more than k, i.e., with $\omega(G) \leq k$, has an upper bound independent of the number of vertices. The best published upper bound is $2^k k^2$ (Kostochka [5]). Recently Kostochka and Kratochvíl [6] proved that the better bound 2^{k+6} is valid even for a wider class of graphs (for the so-called polygon-circle graphs).

The most appealing special case of the problem is the case of triangle-free graphs ($\omega(G) \leq 2$). The lower and upper bounds on the chromatic number of triangle-free circle graphs were first obtained by Karapetyan [4] (1984):

$$4 \leq \chi^* \leq 8;$$

here, and hereafter, $\chi^* = \max\{\chi(G) : G \text{ is a triangle-free circle graph}\}$. He also conjectured that $\chi^* = 4$ (see Problem 11 in [7], or [3] p. 158). A bit later and independently, Gyárfás and Lehel [2] (1985) proved the lower bound $4 \leq \chi^*$ and posed the problem of

*Partially supported by the Russian Foundation for Fundamental Research, grant 93–011–1486

finding the exact value of χ^* (Problem 3.4 in [2]: "Is it true that $\chi^* = 4$?"). The best upper bound is due to Kostochka [5] (1988):

$$\chi^* \leq 5. \tag{1}$$

We show that $\chi^* \geq 5$. This disproves Karapetyan's conjecture and together with (1) gives the exact value

$$\chi^* = 5.$$

The circle graphs considered are assumed to be represented by families of chords of a circle. So we will refer to chords as vertices and vice versa. By a proper k-coloring c of a graph G we mean a mapping $c : V(G) \to \{1, \ldots, k\}$ with the property that $c(v) \neq c(v')$ if $vv' \in E(G)$. Given a circle graph G, we call a vertex subset $X \subseteq V(G)$ *parallel* if X is independent and the graph H with $V(H) = V(G) \cup \{v\}$, $E(H) = E(G) \cup \{xv : x \in X\}$ is also a circle graph.

Theorem. *There exists a triangle-free circle graph with chromatic number equal to* 5.

Proof. We will deduce a contradiction from the opposite assumption that

$$\chi(G) \leq 4 \text{ for each triangle-free circle graph } G. \tag{2}$$

Assume that (2) is true. Then the following two statements must be true.

Claim 1. *For any triangle-free circle graph G and any parallel set $X \subseteq V(G)$, there exists a proper 4-coloring of G such that all vertices in X have the same color.*

Let G be a triangle-free circle graph and $X \subseteq V(G)$ be parallel. Consider the disjoint union $H' = G_0 \cup \ldots \cup G_4$ of five copies of the graph G. Denote by X_i $(i = 0, \ldots, 4)$ the copy of X in G_i. Now let H be the graph obtained from H' by connecting each vertex of X_i $(i = 0, \ldots, 4)$ to each vertex of X_{i+1} where $i + 1$ is taken mod 5. By construction H is triangle-free. A chord intersection representation of H is obtained from that of G and that of the subgraph induced by $X_0 \cup \ldots \cup X_4$. The idea of the construction is clear from Fig. 1, which shows such a representation in the case when $G = K_{1,2}$ and X consists of the two nonadjacent vertices of G. Thus H is a circle triangle-free graph. Hence by (2) there exists a proper 4-coloring c of H. By the construction of H for each $i \in \{0, \ldots, 4\}$, we must have $c(X_i) \cap c(X_{i+1}) = \emptyset$ $(i + 1$ is taken mod 5). It follows that $|c(X_{i'})| = 1$ for some i'. Since $G_{i'}$ is a subgraph of H and a copy of G, the claim is proved.

Consider the circle graph T depicted in Fig. 2 (to save space we present only an essential part of the circle; chords $17, \ldots, 22$ as well as $1, \ldots, 5$ are pairwise non-intersecting). By direct inspection (e. g., using a successive deletion of chords) it is easy to check that T is triangle-free.

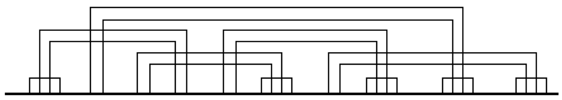

Fig. 1.

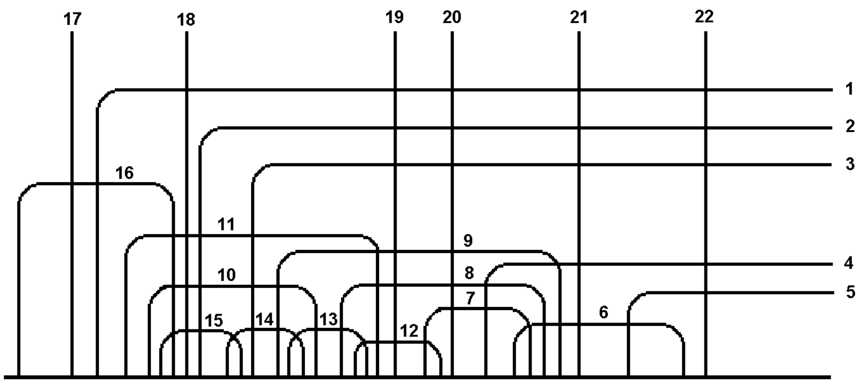

Fig. 2.

Claim 2. *There exists a proper* 4*-coloring* c *of* T *such that* $c(1) = c(2) = \ldots = c(5)$ *and* $c(17) = c(18) = \ldots = c(22)$.

Consider the graph S which is obtained from the disjoint union $T' \cup T''$ of the two copies of T by connecting each vertex in $\{1', \ldots, 5'\}$ with each vertex in $\{1'', \ldots, 5''\}$. It is easy to see that S is a triangle-free circle graph. Furthermore, since the set $\{17', \ldots, 22'\} \cup \{17'', \ldots, 22''\}$ is parallel in S, by Claim 1 there exists a proper 4-coloring c of S such that $c(17') = \ldots = c(22') = c(17'') = \ldots = c(22'')$. By the construction of S it follows that $|c(\{1', \ldots, 5'\})| + |c(\{1'', \ldots, 5''\})| \leq 3$. This proves the claim.

Let c be a proper 4-coloring of T stated in Claim 2. We may assume that $c(1) = c(2) = \ldots = c(5) = 1$, $c(17) = c(18) = \ldots = c(22) = 4$, and $c(6) = 2$. Since chords 7, 8, and 9 intersect chords 4, 6, and 20, we must have $c(7) = c(8) = c(9) = 3$. Furthermore, chords 10 and 11 intersect chords 3, 9, and 18, therefore $c(10) = c(11) = 2$. Since chord 12 intersects chords 7, 11, and 19, we must have $c(12) = 1$. Similarly we successively obtain that necessarily $c(13) = 4$, $c(14) = 2$, and $c(15) = 3$. Now we have: chord 16 intersects chord 1 colored by 1, chord 10 colored by 2, chord 15 colored by 3, and chord 17 colored by 4; a contradiction. The theorem is proved.

Remark. The proof in fact contains a way of constructing an example of the graph with 220 vertices. It is obtained by piecing together five copies of the graph S as described in the proof of Claim 1.

References

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