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DOMINATING SETS AND HAMILTONICITY IN $K_{1,3}$-FREE GRAPHS†)

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1. Introduction. An undirected graph is called $K_{1,3}$-free, if it contains no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$. The class of $K_{1,3}$-free graphs includes all the line graphs and is well studied in many aspects. For instance, it is known that, for $K_{1,3}$-free graphs, the strong Berge conjecture holds [1], and certain problems that are NP-complete in the general setting are polynomially solvable for them [2, 3]. Since the middle of the 70s the articles have begun to appear that are devoted to study of conditions sufficient for hamiltonicity of a $K_{1,3}$-free graph (see Section 3 of the newest survey by Gould [4]). It is the following conjecture extending Thomassen's conjecture for line graphs [5] that became widely known.

Conjecture [6]. *Every* 4*-connected $K_{1,3}$-free graph is Hamiltonian.*

It is known that the Hamiltonian cycle problem is NP-complete for 3-connected cubic planar graphs [7]. Plummer and Pulleyblank [8] showed that there exist infinitely many non-Hamiltonian 3-connected cubic graphs that, moreover, possess no dominating cycles. It is easy to observe that the replacement of vertices with triangles transforms a 3-connected cubic graph into a 3-connected $K_{1,3}$-free graph; moreover, the former is Hamiltonian if and only if such is the latter. Consequently, the above-cited assertions for 3-connected cubic graphs are also valid for the 3-connected $K_{1,3}$-free graphs.

Every Hamiltonian graph is, obviously, 2-connected. In the theorem below, we list some conditions each of which guarantees existence of a Hamiltonian cycle in a 2-connected $K_{1,3}$-free graph.

Theorem 1.1. *Let G be a 2-connected $K_{1,3}$-free graph with n vertices. The graph G is Hamiltonian if at least one of the following conditions holds*:

($c1$) [9] *the subgraph induced by $N(x)$ is connected for every vertex $x \in V(G)$*;

($c2$) [10] $\delta(G) \geq (n-2)/3$;

($c3$) [11] *the diameter of G does not exceed* 2;

($c4$) [12, 13] *for every triple of pairwise nonadjacent vertices $\{x, y, z\} \subseteq V(G)$*,

$$\deg(x) + \deg(y) + \deg(z) \geq n - 2;$$

($c5$) [14] $|N(x) \cup N(y)| \geq (n+1)/3$ *for every pair of vertices $x, y \in V(G)$.*

It is easy to see that each of the conditions ($c1$)–($c5$) of Theorem 1.1 can be expressed as $\forall x R(x)$, where x is a vertex, or a pair, or a triple of vertices and $R(x)$ is some property. In the present article we establish a condition of the form $\exists x R(x)$, where x is a pair of vertices and $R(x)$ is the property of domination.

2. Dominating sets and hamiltonicity. The graphs considered in the article are undirected graphs without loops and multiple edges. Our terminology and notation are mostly borrowed from [5]. Given two vertices v and w in an undirected graph $G = (V, E)$, we denote by uv the edge that connects them. We write $vw \in E$ ($vw \notin E$) when the vertices v and w are adjacent (nonadjacent). Given a vertex $v \in V$, we denote by $N(v)$ the set of all vertices of G adjacent to v. If $X \subseteq V$ then we denote by $G[X]$ the subgraph of G that is induced by the set X. A set of vertices $X \subseteq V$ is called *dominating* if every vertex of the graph is either contained in X or adjacent to some vertex of X. We denote by $(v_1, \dots, v_p)$ $((v_1, \dots, v_p))$ a simple path (cycle) with the set of vertices $\{v_1, \dots, v_p\}$.

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Simple examples demonstrate that, for an arbitrary 2-connected graph, the condition of existence of a vertex adjacent to all other vertices is not sufficient for existence of a Hamiltonian cycle in the graph. As for $K_{1,3}$-free graphs, even the weaker condition of existence of a 2-element dominating set turns out to be sufficient.

Theorem 2.1. *An undirected graph $G = (V, E)$ is Hamiltonian provided that it*
(a) *is 2-connected;*
(b) *is $K_{1,3}$-free;*
(c) *has a 2-element dominating set $\{a, b\} \subseteq V$.*

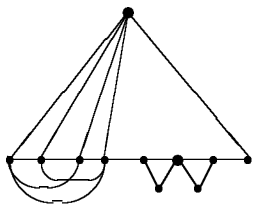

Fig. 1

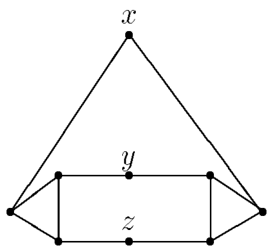

Fig. 2

In Fig. 1 a Hamiltonian 2-connected $K_{1,3}$-free graph is shown which enjoys property (c) but satisfies none of the conditions $(c1)$–$(c5)$. Note also that condition (c) of Theorem 2.1 cannot be weakened: in Fig. 2 an example is presented of a non-Hamiltonian 2-connected $K_{1,3}$-free graph which has no 2-element but has a three-element dominating set, $\{x, y, z\}$.

3. Proof of Theorem 2.1. Suppose G to be non-Hamiltonian. Since G is 2-connected, there is at least one cycle passing through the vertices a and b. Let C be a cycle of maximal length among all such cycles. Since G is non-Hamiltonian, we have $V \setminus V(C) \neq \varnothing$.

Let X stand for the set of vertices of one of the connected components of $G - C$. By (c), we have $X \subseteq N(a) \cup N(b)$; hence $(N(a) \cap X \neq \varnothing) \vee (N(b) \cap X \neq \varnothing)$. Without loss of generality we shall assume that $N(a) \cap X \neq \varnothing$. Then there is a vertex $u \in X$ such that $au \in E$. It follows from (a) that there exists an edge $z\omega \in E$ such that $z \in V(C)$, $\omega \in X$; moreover, $z \neq a$. The cycle C can be expressed in the form

$$C = (a, s_1, \dots, s_k, z, t_l, \dots, t_1, a),$$

where with no loss of generality we may assume that $k \leq l$. Denote by Ω the set of cycles C' such that $\{a, b\} \subseteq V(C')$ and $|V(C')| > |V(C)|$. Note that $\Omega = \varnothing$ by the assumption. Whence we obtain the following:

$$\text{if } v', v'' \in V(C) \text{ and } v'v'' \in E(C), \text{ then } |N(X) \cap \{v', v''\}| \leq 1. \qquad (1)$$

Since the graph $G[X]$ is connected, it contains a simple path with endpoints u and ω. Denote this path by P. In the case when $u = \omega$, we put $P = \{u\}$. If $k = 0$, then

$$(a, t_1, \dots, t_l, z, P, a) \in \Omega;$$

whence $k \geq 1$.

By (1) we have $s_1u, t_1u \notin E$. On the other hand, $s_1a, t_1a, ua \in E$, which implies $s_1t_1 \in E$ by condition (b). A similar consideration gives $a_kt_l \in E$.

If $k = 1$, then

$$(a, s_1, t_1, \dots, t_l, s_k, z, P, a) \in \Omega;$$

and if $k = 2$, then

$$(a, s_1, t_1, \dots, t_l, s_2, z, P, a) \in \Omega.$$

Thus, $k \geq 3$. Now the proof splits into two cases: $N(b) \cap X \neq \varnothing$ and $N(b) \cap X = \varnothing$.

Case 1: $N(b) \cap X \neq \varnothing$. Without loss of generality we may assume that $b = z$. Let $s_1, t_1, \ldots, s_{k-1}t_{k-1} \in E$. Then we have

$$(a, s_1, t_1, t_2, s_2, \ldots, s_{k-1}, t_{k-1}, t_k, \ldots, t_l, s_k, b, P, a) \in \Omega$$

for k even, and

$$(a, t_1, s_1, s_2, t_2, \ldots, s_{k-1}, t_{k-1}, t_k, \ldots, t_l, s_k, b, P, a) \in \Omega$$

for k odd.

Thus, there exists an index $j \leq k-2$ such that

$$s_1t_1, \ldots, s_jt_j \in E, \quad s_{j+1}t_{j+1} \notin E.$$

Now we define simple paths $\mathfrak{P}(a, s_j)$ and $\mathfrak{P}(a, t_j)$ with endpoints a, s_j and a, t_j respectively:

$$\mathfrak{P}(a, s_j) = \begin{cases} (a, s_1, t_1, t_2, s_2, \ldots, t_j, s_j) & \text{if } j \text{ is even,} \\ (a, t_1, s_1, s_2, t_2, \ldots, t_j, s_j) & \text{if } j \text{ is odd;} \end{cases}$$

$$\mathfrak{P}(a, t_j) = \begin{cases} (a, t_1, s_1, s_2, t_2 \ldots, s_j, t_j) & \text{if } j \text{ is even,} \\ (a, s_1, t_1, t_2, s_2, \ldots, s_j, t_j) & \text{if } j \text{ is odd.} \end{cases}$$

If $s_{j+1}b \in E$ then (see Fig. 3)

$$(\mathfrak{P}(a, t_j)t_{j+1}, \ldots, t_l, s_k, \ldots, s_{j+1}, b, P, a) \in \Omega.$$

Similarly, if $t_{j+1}b \in E$ then

$$(\mathfrak{P}(a, s_j), s_{j+1}, \ldots, s_k, t_l, \ldots, t_{j+1}, b, P, a) \in \Omega.$$

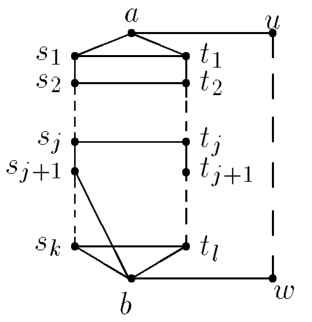

Fig. 3

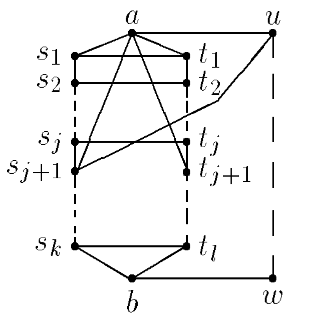

Fig. 4

Thus, $s_{j+1}a, t_{j+1}b \notin E$. However, by (c) we must have $s_{j+1}a, t_{j+1}a \in E$. If $s_{j+1}u \in E$ then (see Fig. 4)

$$(\mathfrak{P}(a, t_j)t_{j+1}, \ldots, t_l, b, s_k, \ldots, s_{j+1}, u, a) \in \Omega.$$

Similarly, if $t_{j+1}u \in E$ then

$$(\mathfrak{P}(a, s_j), s_{j+1}, \ldots, s_k, b, t_l, \ldots, t_{j+1}, u, a) \in \Omega.$$

Thus, $s_{j+1}u, t_{j+1}u \notin E$ and, moreover, $s_{j+1}t_{j+1} \notin E$; $as_{j+1}, at_{j+1}au \in E$. This however means that the set $\{a, s_{j+1}, t_{j+1}, u\}$ induces $K_{1,3}$. We obtain a contradiction.

Case 2: $N(b) \cap X = \varnothing$. Here it is convenient to remove the assumption $k \leq l$; therefore, without loss of generality we may assume that $b \in \{t_1, \ldots, t_l\}, b = t_j$.

If $as_{k-1} \in E$ then

$$(a, s_{k-1}, \ldots, s_1, t_1, \ldots, t_l, s_k, z, P, a) \in \Omega.$$

If $at_{l-1} \in E$ then

$$(at_{l-1}, \dots, t_1, s_1, \dots, s_{k-1}, t_l, z, P, a) \in \Omega.$$

Hence, as_{k-1}, $at_{l-1} \notin E$, which implies

$$(bs_{k-1}, bt_{l-1} \in E) \vee (bs_{k-1} \in E \wedge b = t_{l-1}). \tag{2}$$

If $b = t_1$ then

$$(a, s_1, \dots, s_{k-1}, t_1, \dots, t_l, s_k, z, P, a) \in \Omega.$$

If $b = t_2$ then

$$(a, t_1, s_1, \dots, s_{k-1}, t_2, \dots, t_l, s_k, z, P, a) \in \Omega.$$

Thus, $b \neq t_1, t_2$.

If follows from (2) that there is an index $\mu \in \{1, \dots, k-2\}$ such that $as_\mu \in E$ and $bs_i \in E$ for all $i \in \{\mu+1, \dots, k-1\}$. Now, if $b = t_l$ then

$$(as_\mu, \dots, s_1, t_1, \dots, t_l, s_{\mu+1}, \dots, s_k, z, P, a) \in \Omega.$$

However, if $b = t_{l-1}$ then

$$(as_\mu, \dots, s_1, t_1, \dots, t_{l-1}, s_{\mu+1}, \dots, s_k, t_l, z, P, a) \in \Omega.$$

Thus, we have shown that $3 \leq j \leq l-2$. In view of (2) we now have

$$bt_{j-1}, bs_{\mu+1}, bt_{l-1} \in E. \tag{3}$$

If $s_{\mu+1}t_{j-1} \in E$ then (see Fig. 5)

$$(as_\mu, \dots, s_1, t_1, \dots, t_{j-1}, s_{\mu+1}, \dots, s_{k-1}, b, t_{j+1}, t_l, s_k, z, P, a) \in \Omega.$$

If $t_{j-1}t_{l-1} \in E$ then (see Fig. 6)

$$(as_\mu, \dots, s_1, t_1, \dots, t_{j-1}, t_{l-1}, bs_{\mu+1}, \dots, s_k, t_l, z, P, a) \in \Omega.$$

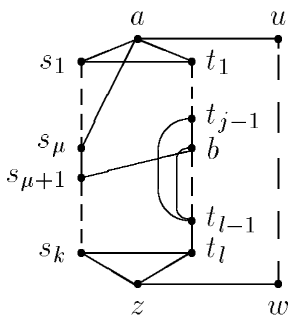

Fig. 5

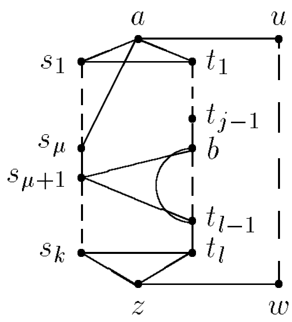

Fig. 6

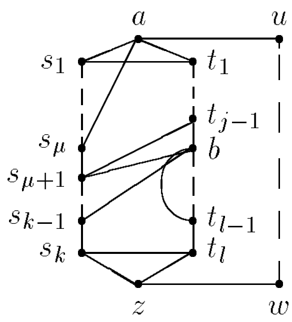

Fig. 7

If $s_{\mu+1}t_{l-1} \in E$ then (see Fig. 7)

$$(as_\mu, \dots, s_1, t_1, \dots, t_{l-1}, s_{\mu+1}, \dots, s_k, t_l, z, P, a) \in \Omega.$$

Thus, $s_{\mu+1}t_{j-1}, s_{\mu+1}t_{l-1}, t_{j-1}t_{l-1} \notin E$. However, it then follows from (3) that the set $\{b, s_{\mu+1}, t_{j-1}, t_{l-1}\}$ induces $K_{1,3}$ in G. We arrive at a contradiction again.

The theorem is proved.

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