ISSN 1990-4789, Journal of Applied and Industrial Mathematics, 2016, Vol. 10, No. 1, pp. 61–68. © Pleiades Publishing, Ltd., 2016.
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A Capacitated Competitive Facility Location Problem

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Received May 20, 2015; in final form, June 22, 2015

Abstract—We consider a mathematical model similar in a sense to competitive location problems. There are two competing parties that sequentially open their facilities aiming to "capture" customers and maximize profit. In our model, we assume that facilities' capacities are bounded. The model is formulated as a bilevel integer mathematical program, and we study the problem of obtaining its optimal (cooperative) solution. It is shown that the problem can be reformulated as that of maximization of a pseudo-Boolean function with the number of arguments equal to the number of places available for facility opening. We propose an algorithm for calculating an upper bound for values that the function takes on subsets which are specified by partial $(0, 1)$-vectors.

DOI: 10.1134/S1990478916010075

Keywords: *bilevel programming, upper bound, competitive facility location*

INTRODUCTION

We study the competitive facilities location problem which differs from the similar problems in [2, 3, 5, 6, 8, 11] by the presence of limited production capacities of the opened facilities. In the competitive location models, unlike the simple facility location problem [1, 14], the two competing parties are considered. These parties sequentially open their facilities to optimize their own objectives. The objectives of parties, which are usually called the *Leader* and the *Follower*, consist in "capturing" the customers and satisfying the customers' demands. At that, the opportunity for one of the parties to capture some customer depends on the customer preferences: these preferences allow to determine which of the parties has opened the most suitable facility for him or her.

The goal of the Leader in this competition is to choose such a set of facilities to be opened which maximizes profit, given the fact that part of the customers will be captured by the Follower. The goal of the Follower is to open such a set of facilities to maximize profit taking into account the knowledge about the facilities opened by the Leader. This interaction of parties in the sequential competitive facility location can be considered as a Stackelberg game [15].

Formally the sequential competitive facility location problem can be represented as an integer bilevel programming problem [7, 9, 10, 12] which includes the upper level (the Leader's) problem and the lower level (the Follower's) problem. The form of these problems depends on the rules of how the parties choose the facility to serve the captured customers and which additional constraints are imposed on the set of opened facilities and on the ability to serve the customers.

The model under study contains the production capacity limitations for the facilities opened by both the Leader and the Follower. This means that each opened facility can be used to serve only the set of customers for which the total customer's demand does not exceed the given production capacity of this facility. With these constraints it is natural to assume that both parties use the rule of free choice of the opened facility to serve the captured customer [3]. Under this assumption the party that has captured the customer can use every open facility which is not worse according to this customer's preferences than each of the facilities opened by the other party to serve this customer.

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The important property of the model under study, as well as for the previously considered competitive facility location models, is the need to define the concept of best solution. This is due to the possible nonuniqueness of the optimal solution of the Follower's problem, which creates the uncertainty in calculation of the value of the Leader's objective function.

For the competitive facility location models considered in [2, 3, 5, 6, 8, 11] the two problems were set: the search for the optimal guaranteed (noncooperative) solution and optimal (cooperative) solution. In this paper we study the problem of finding the optimal solution of the capacitated competitive facility location problem under consideration. This choice is driven by aspiration to use the method for algorithm construction which was developed in [2, 3, 5, 6, 8, 11]. The main idea of this method consists in representation of the investigated problem as a problem of maximizing some pseudo-Boolean function such that the number of variables is equal to the number of potential facility locations of the Leader. In order to implement this representation it is required to show that, for the fixed facility location of the Leader, the corresponding best solution can be obtained as the optimal solution of a certain "regular" integer programming problem whose size is comparable to the size of the Leader's and the Follower's problems.

We show that, in the case of search for the optimal solution, the required pseudo-Boolean function can be constructed. In order to evaluate the values of this function and construct the corresponding optimal solution we have to solve two integer linear programming problems. First of them is the Follower's problem, while the second is an auxiliary problem whose size is equal to the sum of sizes of the Leader's and the Follower's problems.

Another important condition for the implementation of the above-mentioned method is the ability to calculate efficiently the upper bound for the constructed pseudo-Boolean function on the subsets of $(0,1)$-vectors that are defined by the partial $(0,1)$-vectors. In this paper, we provide a way to obtain such an upper bound by using some modified system of estimating subsets [3, 5, 6, 11].

The paper consists of the three sections: In Section 1, we provide the statement of the capacitated competitive facility location problem as a bilevel integer programming problem. In Section 2, we formulate the problem of finding the optimal solution of the model under consideration and provide its reduction to the problem of maximizing the pseudo-Boolean function. In Section 3, we describe a method to calculate an upper bound of the constructed pseudo-Boolean function on the sets of $(0,1)$-vectors defined by the partial $(0,1)$-vectors.

1. STATEMENT OF THE PROBLEM

We introduce the following notions that are needed for the formal statement of the capacitated competitive facility location problem:

As in the simple facility location problem, let $I = \{1, \dots, m\}$ be the set of facilities (or possible facility location places), and $J = \{1, \dots, n\}$, the set of customers.

We assume that the facility $i \in I$ can be opened both by the Leader and the Follower. Therefore, for each $i \in I$, we assume that the values f_i and g_i are equal to the costs to open facility i by the Leader and the Follower correspondingly. If the Leader or the Follower cannot open facility i for some reason, then put $f_i = \infty$ or $g_i = \infty$.

Given $i \in I$ and $j \in J$, let p_{ij} and q_{ij} denote the value of the revenue obtained when serving customer j by facility i opened by the Leader and the Follower correspondingly.

We assume that the capture of customer $j \in J$ by the Leader or the Follower is performed with respect to the preferences of this customer. The preferences of customer $j \in J$ are represented by the order relation $\succ_j$ on I. For $i, k \in I$, $i \succ_j k$ means that customer $j \in J$ would prefer facility i from the two open facilities i and k. The relation $i \succcurlyeq_j k$ means that either $i \succ_j k$ or $i = k$.

Let $I_0 \subset I$. For each $j \in J$, let $i_j(I_0)$ denote $i_0 \in I_0$ such that $i_0 \succcurlyeq_j i$ for every $i \in I_0$. If $I_0 = \{i \in I \mid w_i = 1\}$, where $w = (w_i)$, $i \in I$, is some $(0,1)$-vector, then for $i_j(I_0)$ we will also use the notation $i_j(w)$.

In order to determine which of the parties captures customer $j \in J$, we use the following rule: Let the entries 1 of $(0,1)$-vector $x = (x_i)$, $i \in I$, denote the facilities by the Leader and the entries 1 of $(0,1)$-vector $z = (z_i)$, $i \in I$, denote the facilities by the Follower. We assume that customer $j \in J$ will be captured by the Leader if $i_j(x) \succcurlyeq_j i_j(z)$ and by the Follower if $i_j(z) \succ_j i_j(x)$.

Let the Leader and the Follower use the rule of *free choice* while selecting the facility for the captured customer $j \in J$. This means that if customer j is captured by the Leader then the Leader can choose every open facility $i \in I$ such that $i \succcurlyeq_j i_j(z)$ to serve this customer. Similarly, if customer j is captured by the Follower then the Follower can use each of his/her open facilities $i \in I$ such that $i \succ_j i_j(x)$ to serve this customer.

Unlike for the previously considered competitive facility location models, we assume that the ability for both the Leader's and the Follower's facilities to serve the customers are limited. Let a_{ij} and b_{ij} denote the production volume of facility $i \in I$ opened by the Leader and the Follower correspondingly, required to serve customer $j \in J$. Let V_i and W_i stand for the total capacity of facility $i \in I$ by the Leader and the Follower correspondingly.

The Leader's and Follower's goals are to maximize the total profit which is constituted of the profits of all facilities opened by them, taking into account the production capacities of these facilities. We assume that the profit of every open facility is equal to the sum of profits obtained from all customers served by it, subtracting the fixed cost of opening this facility.

We introduce the following variables similar to those of the simple facility location problem:

$x_i = 1$ if Leader opens facility $i \in I$ and $x_i = 0$ otherwise;

$x_{ij} = 1$ if facility $i \in I$ by the Leader is chosen to serve customer $j \in J$ and $x_{ij} = 0$ otherwise;

$z_i = 1$ if Follower opens the facility $i \in I$ and $z_i = 0$ otherwise;

$z_{ij} = 1$ if facility $i \in I$ opened by Follower is chosen to serve customer $j \in J$, and $z_{ij} = 0$ otherwise.

Using the above variables, we can formulate the mathematical model of the Leader and Follower interaction in the capacitated sequential competitive facility location as the following bilevel integer programming problem:

$$\max_{(x_i),(x_{ij})} \Big\{ -\sum_{i\in I} f_i x_i + \sum_{j\in J}\sum_{i\in I} p_{ij} x_{ij} \Big\}, \tag{1}$$

$$\widetilde{z}_i + \sum_{k | i \succcurlyeq_j k} x_{kj} \leqslant 1, \qquad i \in I, \quad j \in J, \tag{2}$$

$$x_i \geqslant x_{ij}, \qquad i \in I, \quad j \in J, \tag{3}$$

$$\sum_{j\in J} a_{ij} x_{ij} \le V_i, \qquad i \in I, \tag{4}$$

$$x_i, x_{ij} \in \{0,1\}, \qquad i \in I, \quad j \in J, \tag{5}$$

$$(\widetilde{z}_i),\ (\widetilde{z}_{ij}) \ \text{ is the optimal solution of the problem,} \tag{6}$$

$$\max_{(z_i),(z_{ij})} \Big\{ -\sum_{i\in I} g_i z_i + \sum_{j\in J}\sum_{i\in I} q_{ij} z_{ij} \Big\}, \tag{7}$$

$$x_i + z_i \le 1, \qquad i \in I, \tag{8}$$

$$x_i + \sum_{k | i \succcurlyeq_j k} z_{kj} \leqslant 1, \qquad i \in I, \quad j \in J, \tag{9}$$

$$z_i \geqslant z_{ij}, \qquad i \in I, \quad j \in J, \tag{10}$$

$$\sum_{j\in J} b_{ij} z_{ij} \le W_i, \qquad i \in I, \tag{11}$$

$$z_i, z_{ij} \in \{0,1\}, \qquad i \in I, \quad j \in J. \tag{12}$$

The upper level problem (1)–(5) is called *Problem* $\mathcal{L}$, while the lower level problem (7)–(12), *Problem* $\mathcal{F}$. Problem (1)–(12) as a whole is *Problem* $(\mathcal{L}, \mathcal{F})$, while the objective function (1) of Problem $\mathcal{L}$ is assumed to be the objective function of Problem $(\mathcal{L}, \mathcal{F})$.

The objective function (1) of Problem $\mathcal{L}$ represents the profit by the Leader. Inequalities (2) forbid to serve the customer by those Leader's facilities that are less preferable for this customer than some of the facilities opened by the Follower. These inequalities also guarantee that each of the customers can be served with at most one Leader's facility. Constraint (3) ensures that the serving the customers can be performed only by the open facility. Condition (4) guarantees that the total production demand satisfied by each of the opened facilities does not exceed its production capacity. The objective function and constraints of Problem $\mathcal{F}$ have similar meaning. Additional constraint (8) shows that the Follower cannot open the facility in the location where the Leader had already opened a facility.

2. OPTIMAL SOLUTIONS OF PROBLEM $(\mathcal{L}, \mathcal{F})$

Let us refer to a pair $(X, \widetilde{Z})$ as a *feasible solution* of Problem $(\mathcal{L}, \mathcal{F})$ if $X = ((x_i), (x_{ij}))$ is a feasible solution of Problem $\mathcal{L}$ for a given vector $(\widetilde{z}_i)$ and $\widetilde{Z} = ((\widetilde{z}_i), (\widetilde{z}_{ij}))$ is an optimal solution of Problem $\mathcal{F}$ for a given vector (x_i),

Let $L(X, \widetilde{Z})$ denote the value of the objective function of Problem $(\mathcal{L}, \mathcal{F})$ at a feasible solution $(X, \widetilde{Z})$, and let $F(Z)$ denote the value of the objective function of Problem $\mathcal{F}$ at a feasible solution Z.

A feasible solution (X^*, Z^*) is called an *optimal solution* of Problem $(\mathcal{L}, \mathcal{F})$ if $L(X^*, Z^*) \geq L(X, \widetilde{Z})$ for every feasible solution $(X, \widetilde{Z})$.

Then, we focus on the problem of finding an optimal solution for the $(\mathcal{L}, \mathcal{F})$ model. Since, given some fixed feasible solution X of Problem $\mathcal{L}$, an optimal solution $\widetilde{Z}$ of Problem $\mathcal{F}$ is generally not unique, and, for different solutions $\widetilde{Z}_1$ and $\widetilde{Z}_2$, the values $L(X, \widetilde{Z}_1)$ and $L(X, \widetilde{Z}_2)$ can differ; therefore, the current problem is stated as follows:

Find

$$\max_{(x_i),(x_{ij})} \max_{(\widetilde{z}_i),(\widetilde{z}_{ij})} \Big\{ -\sum_{i\in I} f_i x_i + \sum_{i\in J}\sum_{i\in I} p_{ij} x_{ij} \Big\}$$

satisfying (2)–(12). We denote this problem by $(\mathcal{L}, \mathcal{F})$ too.

Note that for the fixed $(0,1)$-vector $x = (x_i)$, $i \in I$, the corresponding optimal solution (X^*, Z^*) of Problem $(\mathcal{L}, \mathcal{F})$, where $X^* = ((x_i), (x^*_{ij}))$, can be obtained by the algorithm which consists of the two stages:

At the first stage, for the fixed $(0,1)$-vector $x = (x_i)$, $i \in I$, Problem $\mathcal{F}$ is solved and the optimal value of its objective function F^* is determined.

At the second stage, the auxiliary problem is solved:

Find

$$\max_{(x_{ij})} \max_{(z_i),(z_{ij})} \Big\{ -\sum_{i\in I} f_i x_i + \sum_{i\in J}\sum_{i\in I} p_{ij} x_{ij} \Big\}$$

satisfying

$$z_i + \sum_{k|i \succcurlyeq_j k} x_{kj} \leq 1, \qquad i \in I, \quad j \in J,$$

$$x_i \geq x_{ij}, \qquad i \in I, \quad j \in J,$$

$$\sum_{j\in J} a_{ij} x_{ij} \leq V_i, \qquad i \in I,$$

$$-\sum_{i\in I} g_i z_i + \sum_{j\in J}\sum_{i\in I} q_{ij} z_{ij} \geq F^*,$$

$$x_i + z_i \leq 1, \qquad i \in I,$$

$$x_i + \sum_{k|i \succcurlyeq_j k} z_{kj} \leq 1, \qquad i \in I, \quad j \in J,$$

$$z_i \geq z_{ij}, \qquad i \in I, \quad j \in J,$$

$$\sum_{j\in J} b_{ij} z_{ij} \leq W_i, \qquad i \in I,$$

$$x_{ij}, z_i, z_{ij} \in \{0,1\}, \qquad i \in I, \quad j \in J.$$

It is clear that if $((x^*_{ij}), (z^*_i), (z^*_{ij}))$ is an optimal solution for this problem then $Z^* = ((z^*_i), (z^*_{ij}))$ is an optimal solution of Problem $\mathcal{F}$, and the feasible solution (X^*, Z^*), $X^* = ((x_i), (x^*_{ij}))$, is an optimal solution of Problem $(\mathcal{L}, \mathcal{F})$ for the fixed vector $x = (x_i),\ i \in I$.

This implies that Problem $(\mathcal{L}, \mathcal{F})$ can be represented as that of maximizing some pseudo-Boolean function $f(x)$. The value of this function on the $(0, 1)$-vector x is the optimal value of the objective function of the auxiliary problem under consideration. Hence, to calculate the value of $f(x)$ at a vector x, we have to solve Problem $\mathcal{F}$, and then solve the described auxiliary problem. In result we find both the value of $f(x)$ at (0,1)-vector x, and the corresponding feasible solution of Problem $(\mathcal{L}, \mathcal{F})$.

3. AN UPPER BOUND

Let us address the problem of efficient calculation of an upper bound for the values of the pseudo-Boolean function $f(x)$ under consideration, $x = (x_i)$, $i \in I$, for the subsets of the set of $(0, 1)$-vectors. These subsets can be conveniently given using partial $(0, 1)$-vectors. We call the vector $y = (y_i)$, $i \in I$, whose entries take the values 0, 1, and the undefined value $*$, a *partial* $(0, 1)$-*vector* or a *partial solution*. A partial solution splits the variables of $f(x)$ into the variables with the value 0 or 1 and free variables. Given the partial $(0, 1)$-vector $y = (y_i)$, we define $I^0 = \{i \in I \mid y_i = 0\}$ and $I^1 = \{i \in I \mid y_i = 1\}$. The partial solution $y = (y_i)$ defines the set of $(0, 1)$-vectors $x = (x_i)$ such that $x_i = 0$ for $i \in I^0$ and $x_i = 1$ for $i \in I^1$. Let this set of vectors be denoted by $P(y)$.

The value of $f(x)$ at a $(0, 1)$-vector $x \in P(y)$ is the value of the objective function of Problem $(\mathcal{L}, \mathcal{F})$ at an optimal solution (X^*, Z^*) corresponding to x. Let $y = (y_i),\ i \in I$, be a partial $(0, 1)$-vector for which $I^0(y) \cup I^1(y) \neq I$.

The maximal value of $f(x)$ on $P(y)$ is the optimal value of the objective function of Problem $(\mathcal{L}, \mathcal{F})$ with the additional constraint $x_i = y_i$ for $i \in I^0(y) \cup I^1(y)$. This problem can be stated as follows:

Find

$$\max_{(x_i),(x_{ij})} \max_{(\widetilde{z}_i),(\widetilde{z}_{ij})} \Big\{ -\sum_{i\in I} f_i x_i + \sum_{j\in J}\sum_{i\in I} p_{ij} x_{ij} \Big\}, \tag{13}$$

satisfying

$$\widetilde{z}_i + \sum_{k \mid i \succ_j k} x_{kj} \leqslant 1, \qquad i \in I, \quad j \in J, \tag{14}$$

$$x_i \geqslant x_{ij}, \qquad i \in I, \quad j \in J, \tag{15}$$

$$\sum_{j\in J} a_{ij} x_{ij} \leq V_i, \qquad i \in I, \tag{16}$$

$$x_i = y_i, \qquad i \in I^0(y) \cup I^1(y), \tag{17}$$

$$x_i, x_{ij} \in \{0, 1\}, \qquad i \in I, \quad j \in J, \tag{18}$$

$$(\widetilde{z}_i), (\widetilde{z}_{ij}) \text{ is the optimal solution of (7)–(12).} \tag{19}$$

Problem (13)–(18) is denoted by *Problem* $\mathcal{L}(y)$, and the problem (13)–(19), (7)–(12) as a whole, by *Problem* $(\mathcal{L}(y), \mathcal{F})$. The objective function of Problem $(\mathcal{L}(y), \mathcal{F})$ is denoted by $L(X, \widetilde{Z})$.

Let us modify the method for constructing the system of estimating subsets $\{I_j(y)\},\ j \in J$, from [5, 6, 11] to apply it to Problem $(\mathcal{L}'(y), \mathcal{F})$. We construct the estimating problem such that the optimal value of its objective function will provide an upper bound for the values of the objective function of Problem $(\mathcal{L}'(y), \mathcal{F})$.

For a given partial $(0, 1)$-vector $y = (y_i),\ i \in I$, and fixed $j_0 \in J$ we formulate the rules that allow us to determine whether $i \in I_{j_0}(y)$ or $i \notin I_{j_0}(y)$ for each $i \in I$.

If $y_i = 0$ then $i \notin I_{j_0}(y)$. Let $y_i \neq 0$. Consider the set $N(i) = \{k \in I \mid k \succ_{j_0} i\}$. If $N(i) = \varnothing$ then $i \in I_{j_0}(y)$. Assume that $N(i) \neq \varnothing$. If $N(i) \cap I^1(y) \neq \varnothing$ then $i \notin I_{j_0}(y)$.

Let $N(i) \cap I^1(y) = \varnothing$. Consider

$$J(i) = \{j \in J \mid \text{ if } k \succ_j i_j(I^1(y) \cup \{i\}) \text{ then } k \in N(i)\}.$$

Note that $J(i) \neq \varnothing$ since $j_0 \in J(i)$. Given $k \in N(i)$, we consider the set

$$J(k, i) = \{j \in J(i) \mid k \succ_j i_j(I^1(y) \cup \{i\})\}.$$

We assume that $i \notin I_{j_0}(y)$ if for some $k \in N(i)$ there exists $S(k) \subset J(k, i)$ such that

$$g_k < \sum_{j \in S(k)} q_{kj}, \qquad W_k \geq \sum_{j \in S(k)} b_{kj}.$$

If there is no $k \in N(i)$ with the property then $i \in I_{j_0}(y)$.

The following provides the main property of the estimating subsets:

Lemma 1. *Let $(X, \widetilde{Z})$, where $X = ((x_i), (x_{ij}))$ and $\widetilde{Z} = ((\widetilde{z}_i), (\widetilde{z}_{ij}))$, be a feasible solution of Problem $(\mathcal{L}'(y), \mathcal{F})$, and let $\{I_j(y)\}$, $j \in J$, be a system of estimating subsets. Then, for each $j_0 \in J$, if $i_{j_0}(x) \notin I_{j_0}(y)$, where $x = (x_i)$, $i \in I$, then $\sum_{i \in I} x_{ij_0} = 0$.*

Proof. Consider the $(0, 1)$-vectors $x = (x_i)$, $i \in I$, and $\widetilde{z} = (\widetilde{z}_{ij})$, $i \in I$. Assume that $i_0 = i_{j_0}(x)$ and $i_j = i_j(I^1(x) \cup I^1(\widetilde{z}))$, $j \in J$. Put $N(i_0) = \{k \in I \mid k \succ_{j_0} i_0\}$. Since $i_0 \notin I_{j_0}(y)$, $N(i_0)$ is nonempty. Note that $x_i = 0$ for all $i \in N(i_0)$. Note also that if $\widetilde{z}_i \neq 0$ for some $i \in N(i_0)$ then $\sum_{i \in I} x_{ij_0} = 0$ due to constraints (14) and (15).

Let $\widetilde{z}_i = 0$ for $i \in N(i_0)$. Consider $J(i_0) = \{j \in J \mid \text{if } k \succ_j i_j(I^1(y) \cup \{i_0\}),\ \text{then } k \in N(i_0)\}$. Since $x_i = \widetilde{z}_i = 0$ for $i \in N(i_0)$, we have $i_j = i_j(I^1(y) \cup \{i_0\})$ for each $j \in J(i_0)$.

Since $i_0 \notin I_{j_0}(y)$, there exist $k \in N(i_0)$ and $S(k) \subset \{j \in J(i_0) \mid k \succ_j i_j(I^1(y) \cup \{i_0\})\}$ such that

$$g_k < \sum_{j \in S(k)} q_{kj}, \qquad W_k \geq \sum_{j \in S(k)} b_{kj}.$$

We construct a feasible solution $Z = ((z_i), (z_{ij}))$ of Problem $\mathcal{F}$ which differs from the optimal solution $\widetilde{Z}$ only by $z_k = 1$ and $z_{kj} = 1$ for $j \in S(k)$. For the solutions Z and $\widetilde{Z}$ we have

$$F(Z) - F(\widetilde{Z}) = -g_k + \sum_{j \in S(k)} q_{kj} > 0.$$

This contradicts the fact that $\widetilde{Z}$ is an optimal solution of Problem $\mathcal{F}$.

The proof of Lemma 1 is complete. □

In order to calculate an upper bound of the values of the objective function of Problem $(\mathcal{L}(y), \mathcal{F})$, we construct the auxiliary problem by adding to Problem $\mathcal{L}(y)$ the dummy variables $t_{ij} \in \{0, 1\}$, $i \in I$, $j \in J$, and additional constraints.

This auxiliary problem is stated as follows:

Find

$$\max_{(x_i), (x_{ij}), (t_{ij})} \max_{(\widetilde{z}_i), (\widetilde{z}_{ij})} \Big\{ -\sum_{i \in I} f_i x_i + \sum_{j \in J} \sum_{i \in I} p_{ij} x_{ij} \Big\} \tag{20}$$

satisfying

$$\widetilde{z}_i + \sum_{k | i \succcurlyeq_j k} x_{kj} \leq 1, \qquad i \in I, \quad j \in J, \tag{21}$$

$$x_i \geq x_{ij}, \qquad i \in I, \quad j \in J, \tag{22}$$

$$\sum_{j \in J} a_{ij} x_{ij} \leq V_i, \qquad i \in I, \tag{23}$$

$$x_i + \sum_{k | i \succ_j k} t_{kj} \leq 1, \qquad i \in I, \quad j \in J, \tag{24}$$

$$x_i \geq t_{ij}, \qquad i \in I, \quad j \in J, \tag{25}$$

$$\sum_{i \in I} t_{ij} = 1, \qquad j \in J, \tag{26}$$

$$\sum_{i \in I} x_{ij} \leq \sum_{i \in I_j} t_{ij}, \qquad j \in J, \tag{27}$$

$$x_i = y_i, \qquad i \in I^0(y) \cup I^1(y), \tag{28}$$

$$x_i, x_{ij}, t_{ij} \in \{0, 1\}, \qquad i \in I, \quad j \in J, \tag{29}$$

$$(\widetilde{z}_i), (\widetilde{z}_{ij}) \text{ is the optimal solution of Problem (7)}-(12). \tag{30}$$

Let problem (20)–(30) be called *Problem* $\mathcal{L}'(y)$; and let the whole problem (20)–(30), (7)–(12) be called *Problem* $(\mathcal{L}'(y), \mathcal{F})$. Let $L'(X, T, \widetilde{Z})$ be the value of (20) at a feasible solution $(X, T, \widetilde{Z})$, where $X = ((x_i), (x_{ij}))$, $T = (t_{ij})$, and $\widetilde{Z} = ((\widetilde{z}_i), (\widetilde{z}_{ij}))$.

Note that if (X, T), $X = ((x_i), (x_{ij}))$ and $T = (t_{ij})$, is the feasible solution of Problem $\mathcal{L}'(y)$ then, by (24)–(26), for all $i \in I$ and $j \in J$, we have for $x = (x_i),\ i \in I$,

$$t_{ij} = \begin{cases} 1, & \text{if } i = i_j(x), \\ 0, & \text{otherwise.} \end{cases}$$

We will also consider the following problem called the *estimating* problem for Problem $(\mathcal{L}'(y), \mathcal{F})$; it is obtained from Problem $\mathcal{L}'(y)$ by excluding the constraints that contain the optimal values of the variables of Problem $\mathcal{F}$:
Find

$$\max_{(x_i),(x_{ij}),(t_{ij})} \Big\{ -\sum_{i \in I} f_i x_i + \sum_{j \in J} \sum_{i \in I} p_{ij} x_{ij} \Big\},$$

$$x_i \geq x_{ij}, \qquad i \in I, \quad j \in J,$$

$$\sum_{j \in J} a_{ij} x_{ij} \leq V_i, \qquad i \in I,$$

$$x_i + \sum_{k | i \succ_j k} t_{kj} \leq 1, \qquad i \in I, \quad j \in J,$$

$$x_i \geq t_{ij}, \qquad i \in I, \quad j \in J,$$

$$\sum_{i \in I} t_{ij} = 1, \qquad j \in J,$$

$$\sum_{i \in I} x_{ij} \leq \sum_{i \in I_j(y)} t_{ij}, \qquad j \in J,$$

$$x_i = y_i, \qquad i \in I^0(y) \cup I^1(y),$$

$$x_i, x_{ij}, t_{ij} \in \{0, 1\}, \qquad i \in I, \quad j \in J.$$

Let $B(X, T)$ be the value of the objective function of this problem at a feasible solution (X, T), where $X = ((x_i), (x_{ij}))$ and $T = (t_{ij})$; and let (X^0, T^0), $X^0 = \big((x_i^0), (x_{ij}^0)\big)$ and $T^0 = \big(t_{ij}^0\big)$, denote the optimal solution of the estimating problem.

Theorem. *If* $(X, \widetilde{Z})$ *is a feasible solution of Problem* $(\mathcal{L}'(y), \mathcal{F})$ *then* $L(X, \widetilde{Z}) \leq B(X^0, T^0)$.

Proof. Using the feasible solution $(X, \widetilde{Z})$, where $X = ((x_i), (x_{ij}))$ and $\widetilde{Z} = ((\widetilde{z}_i), (\widetilde{z}_{ij}))$, we construct the solution $(X, T, \widetilde{Z})$ of Problem $(\mathcal{L}'(y), \mathcal{F})$ by putting for $x = (x_i),\ i \in I$,

$$t_{ij} = \begin{cases} 1, & \text{if } i = i_j(x), \\ 0, & \text{otherwise,} \end{cases}, \qquad i \in I, \qquad j \in J.$$

Note that $(X, T, \widetilde{Z})$ is a feasible solution of Problem $(\mathcal{L}'(y), \mathcal{F})$. Indeed, constraints (24)–(26) hold by construction and (27) holds by Lemma 1. Note also that the values of the objective functions of the above problems at the solution $(X, \widetilde{Z})$ and the corresponding solution $(X, T, \widetilde{Z})$ are equal.

Moreover, note that if $(X, T, \widetilde{Z})$ is a feasible solution of Problem $(\mathcal{L}'(y), \mathcal{F})$ then (X, T) is a feasible solution of the estimating problem and the values of the objective functions of the problems are equal.

So $L(X, \widetilde{Z}) = L'(X, T, \widetilde{Z}) = B(X, T) \leq B(X^0, T^0)$, which completes the proof of the theorem. □

Thus, the calculation of the upper bound of the values of constructed pseudo-Boolean function $f(x)$ on $P(y)$, where $y = (y_i)$, $i \in I$, is a partial (0,1)-vector, for which $I^0(y) \cup I^1(y) \neq I$, is reduced to finding the optimal value of the objective function of the estimating problem.

Representation of the problem of finding an optimal solution in the capacitated competitive facility location model as the problem of maximization of certain pseudo-Boolean function of the same variables, as for other variants of competitive facility location problem [2, 3, 5, 6, 8, 11], allows us to use the approved set of algorithms based on the local search approach [2, 5, 12, 13]. The demonstrated ability to efficiently calculate the upper bounds of the values of the considered pseudo-Boolean function on the solution subsets defined by the partial $(0, 1)$-vectors allows us to construct the branch-and-bound algorithms for the problem under consideration which are similar to the algorithms for the problems from [1, 6, 7].

ACKNOWLEDGMENTS

The authors were supported by the Russian Foundation for Basic Research (project no. 15–01–01446).

REFERENCES

1. V. L. Beresnev, *Discrete Location Problems and Polynomials of Boolean Variables* (Inst. Mat., Novosibirsk, 2005) [in Russian].
2. V. L. Beresnev, "Local Search Algorithms for the Problem of Competitive Location of Enterprises," Avtomat. i Telemekh. No. 3, 12–27 (2012) [Automat. Remote Control **73** (3), 425–439 (2012)].
3. V. L. Beresnev, "On the Competitive Facility Location Problem with a Free Choice of Suppliers," Avtomat. i Telemekh. No. 4, 94–105 (2014) [Automat. Remote Control **75** (4), 668–676 (2014)].
4. V. L. Beresnev, E. N. Goncharov, and A. A. Mel'nikov, "Local Search with a Generalized Neighborhood in the Optimization Problem for Pseudo-Boolean Functions," Diskret. Anal. Issled. Oper. **18** (4), 3–16 (2011) [J. Appl. Indust. Math. **6** (1), 22–30 (2012)].
5. V. L. Beresnev and A. A. Mel'nikov, "Approximate Algorithms for the Competitive Facility Location Problem," Diskret. Anal. Issled. Oper. **17** (6), 3–19 (2010) [J. Appl. Indust. Math. **5** (2), 180–190 (2011)].
6. V. L. Beresnev and A. A. Mel'nikov, "The Branch-and-Bound Algorithm for a Competitive Facility Location Problem with the Prescribed Choice of Suppliers," Diskret. Anal. Issled. Oper. **21** (2), 3–23 (2014) [J. Appl. Indust. Math. **8** (2), 177–189 (2014)].
7. A. V. Kononov, Yu. A. Kochetov, and A. V. Plyasunov, "Competitive Facility Location Models," Zh. Vychisl. Mat. Mat. Fiz. **49** (6), 1037–1054 (2009) [Comput. Math. Math. Phys. **49** (6), 994–1009 (2009)].
8. A. A. Mel'nikov, "Randomized Local Search for the Discrete Competitive Facility Location Problem," Avtomat. i Telemekh. No. 4, 134–152 (2014) [Automat. Remote Control **75** (4), 700–714 (2014)].
9. A. V. Plyasunov and A. A. Panin, "The Pricing Problem. Part I: Exact and Approximate Algorithms," Diskret. Anal. Issled. Oper. **19** (5), 83–100 (2012) [J. Appl. Indust. Math. **7** (2), 241-251 (2013)].
10. A. V. Plyasunov and A. A. Panin, "The Pricing Problem. Part II: Computational Complexity," Diskret. Anal. Issled. Oper. **19** (6), 56–71 (2012) [J. Appl. Indust. Math. **7** (3), 420-430 (2013)].
11. V. L. Beresnev, "Branch-and-Bound Algorithm for a Competitive Facility Location Problem," Comput. Oper. Res. **40** (8), 2062–2070 (2013).
12. S. Dempe, *Foundations of Bilevel Programming* (Kluwer Acad. Publ., Dordrecht, 2002).
13. P. L. Hammer and S. Rudeanu, "Pseudo-Boolean Programming," Oper. Res. **17** (2), 233–261 (1969).
14. J. Krarup and P. M. Pruzan, "The Simple Plant Location Problem: Survey and Synthesis," European J. Oper. Res. **12** (1), 36–81 (1983).
15. H. von Stackelberg, *The Theory of the Market Economy* (Hedge, London, 1952).