ISSN 1990-4789, Journal of Applied and Industrial Mathematics, 2017, Vol. 11, No. 4, pp. 472–480. © Pleiades Publishing, Ltd., 2017.
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# An Upper Bound for the Competitive Location and Capacity Choice Problem with Multiple Demand Scenarios

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Received May 2, 2017

**Abstract**—A new mathematical model is considered related to competitive location problems where two competing parties, the Leader and the Follower, successively open their facilities and try to win customers. In the model, we consider a situation of several alternative demand scenarios which differ by the composition of customers and their preferences. We assume that the costs of opening a facility depend on its capacity; therefore, the Leader, making decisions on the placement of facilities, must determine their capacities taking into account all possible demand scenarios and the response of the Follower. For the bilevel model suggested, a problem of finding an optimistic optimal solution is formulated. We show that this problem can be represented as a problem of maximizing a pseudo-Boolean function with the number of variables equal to the number of possible locations of the Leader's facilities. We propose a novel system of estimating the subsets that allows us to supplement the estimating problems, used to calculate the upper bounds for the constructed pseudo-Boolean function, with additional constraints which improve the upper bounds.

**DOI:** 10.1134/S1990478917040020

Keywords: *competitive facility location, bilevel programming, upper bound*

## INTRODUCTION

We consider a mathematical model from the family of models of competitive facility location, which generalizes the classical problem of facility location built on the basis of the idea of the Stackelberg game [12] and written as a bilevel mathematical programming problem [7]. Some surveys of the research aimed at studying these models are presented in [4, 9, 11]. It is assumed that there are two competing parties successively opening their facilities with the aim of capture consumers and making maximum profit. A party, called *the Leader*, opens its facilities first, and then the other party, called *the Follower*, opens its facilities, knowing the decision made by the Leader. Capturing a consumer by one of the parties depends on the preferences of this consumer, represented as a linear order on the set of possible locations of facilities. The consumer is captured by the party that opened the facility most preferable for this consumer. In addition, the party that has captured the consumer can use for serving him/her only those opened facilities that are more preferable for this consumer than any facility opened by the other party. The task of the Leader under conditions of such a competition is to determine a set of the facilities opened by it, which allows the Leader to get the maximum profit under the optimal behavior of the Follower also seeking to maximize the profit.

In the model under study, we consider a situation where several alternative demand scenarios are possible; i.e., alternative sets of consumers with the corresponding preferences, of which only one will be realized. The scenario becomes known after the Leader places its facilities and before the Follower's making a decision on the placement of facilities. Thus, the Follower, when deciding to open its facilities, knows not only the location of the Leader's facilities, but also the composition of consumers and their preferences. In this situation, the task of the Leader is to determine the set of facilities which gives the

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greatest profit in the case of realization of the worst-case scenario of demand and the optimal location of the Follower's facilities.

In the situation of alternative demand scenarios, in the case when a more powerful facility requires more expenses for its opening, the question arises about a choice of capacities of the facilities being opened [10]. In the model under study, it is assumed that the costs of opening facilities are fixed costs plus the costs proportional to the planned volume of production. Note that the problem of choosing the capacity of new facilities can be formulated both for the Leader and the Follower. However, it is essential only for the Leader who does not know which consumers will be served by the facility being opened. In the case of the Follower, the variable costs for opening a facility can be accounted for when determining the amount of income received by the facility from serving each consumer.

An important feature of the proposed model, as well as of the earlier considered models of competitive facility location [1, 2, 5], is the need to clarify the concept of optimal solution. This is due to the possible nonuniqueness of the optimal solution of the Follower, which creates uncertainty when calculating the Leader's objective function. For the model under consideration, a problem is formulated of finding an optimistic optimal solution [7]. The proposed approach to the construction of algorithms for the exact and approximate solutions of this problem is based on the method used in [2, 5] for solving other problems of competitive facility location. The main idea of this method is to represent the problem under study in the form of a problem of maximizing a certain pseudo-Boolean function with the number of variables equal to the number of possible locations for the Leader's facilities. If such a representation is obtained then a possibility arises to construct the effective approximate algorithms based on local search methods and various metaheuristics [3, 6, 8, 13]. Another important component of this method is the calculation of the upper bounds for the values of the constructed pseudo-Boolean function on the subsets of $(0, 1)$-vectors given by partial $(0, 1)$-vectors. A successful solution of this problem allows us to construct the exact algorithms based on the schemes of implicit enumeration.

In this article, we further develop the method for constructing estimating problems to calculate the upper bounds in the problems of competitive location of facilities. The method consists in forming some additional constraints in the Leader problem, not changing the optimal value of its objective function. The technique for constructing such constraints is developed in [1, 2] and is based on the use of so-called estimating subsets. In this article we construct a new system of estimating subsets, which allows us to supplement the estimating problems with new constraints improving the accuracy of the obtained upper bounds.

The article consists of four sections: In Section 1 we construct a mathematical model of competitive facility location and choice of their capacities under multiple demand scenarios. In Section 2 we formulate a problem of choosing an optimistic optimal solution of this model and carry out its reduction to the problem of maximization of a pseudo-Boolean function. Section 3 is devoted to the construction of systems of estimating subsets on the basis of which we formulate the estimating problems for calculating the upper bounds of the pseudo-Boolean function on the subsets formed by partial $(0, 1)$-vectors.

## 1. MATHEMATICAL MODEL

Let us introduce the notation.

For the sets:

$I = \{1, \dots, m\}$ is the set of facilities (possible locations of facilities),

$S = \{1, \dots, l\}$ is the set of possible demand scenarios,

$J_s$ is the set of consumers in the case of realization of the scenario $s \in S$.

We assume that $J_{s_1} \cap J_{s_2} = \varnothing$ for every $s_1, s_2 \in S$, $s_1 \neq s_2$. The set of all possible consumers is denoted by $J = \bigcup_{s \in S} J_s$; we assume that $J = \{1, \dots, n\}$.

For the parameters:

$f_i$ are fixed costs for the Leader's opening the facility $i \in I$,

$c_i$ are unit costs for installing capacities in the facility $i \in I$ opened by the Leader,

$g_i$ are fixed costs for the Follower's opening the facility $i \in I$,

$p_{ij}$ is the income received by the Leader's facility $i \in I$ from the consumer $j \in J$,

$q_{ij}$ is the income received by the Follower's facility $i \in I$ from the consumer $j \in J$,

$a_{ij}$ is the volume of production of the facility $i \in I$ opened by the Leader, necessary for serving the customer $j \in J$.

For the variables:
$x_i$ is a variable equal to one if the Leader opens the facility $i \in I$, and zero, otherwise;
$x_{ij}$ is a variable equal to the share of the demand of the consumer $j \in J$ satisfied by the Leader's facility $i \in I$,
$v_i$ is the production volume (capacity) of the Leader's facility $i \in I$,
$z_i^s$ is a variable equal to one if the Follower opens the facility $i \in I$, and zero, otherwise,
$z_{ij}$ is a variable equal to one if the Follower's facility $i \in I$ is assigned for servicing the consumer $j \in J$, and zero, otherwise.

We assume that the preferences of the consumer $j \in J$ are given by a linear order $\succcurlyeq_j$ on $I$. For $i_1, i_2 \in I$ the relation $i_1 \succcurlyeq_j i_2$ means that either $i_1 = i_2$ or for the consumer $j \in J$ the facility $i_1$ is more preferable as compared to $i_2$. In the case when $i_1 \neq i_2$ and $i_1 \succcurlyeq_j i_2$, we use the notation $i_1 \succ_j i_2$. Given $j \in J$, we denote by $N_j(i)$ the set of elements $\{k \in I \mid k \succ_j i\}$ that are more preferable in terms of the order $\succcurlyeq_j$ as compared with the element $i \in I$. We denote by $\alpha_j(I')$ and $\omega_j(I')$ the maximum and minimum elements of a nonempty subset $I' \subseteq I$ with respect to the order $\succcurlyeq_j$, respectively. Given nonzero $(0,1)$-vector $x = (x_i)$, $i \in I$, we assume that $\alpha_j(x) = \alpha_j(\{i \in I \mid x_i = 1\})$.

Using the above notation, we write the model of competitive facility location and choice of the capacities under multiple demand scenarios as the model of bilevel programming:

$$\max_{(x_i),(x_{ij}),(v_i)} \Big( -\sum_{i\in I}(f_i x_i + c_i v_i) + \min_{s\in S} \sum_{j\in J_s} \sum_{i\in I} p_{ij} x_{ij} \Big), \tag{1}$$

$$\tilde{z}_i^s + \sum_{k \mid i \succcurlyeq_j k} x_{kj} \leqslant 1, \qquad i \in I, \quad s \in S, \quad j \in J_s, \tag{2}$$

$$x_i \geqslant x_{ij}, \qquad i \in I, \quad j \in J, \tag{3}$$

$$\sum_{j\in J_s} a_{ij} x_{ij} \le v_i, \qquad i \in I, \quad s \in S, \tag{4}$$

$$x_i \in \{0,1\}, \qquad x_{ij} \in [0,1], \qquad v_i \geq 0, \qquad i \in I, \quad j \in J, \tag{5}$$

$$\big(\tilde{z}_i^s\big), (\tilde{z}_{ij}) \text{ is the optimal solution of the problem:} \tag{6}$$

$$\max_{(z_i^s),(z_{ij})} \sum_{s\in S} \Big( -\sum_{i\in I} g_i z_i^s + \sum_{j\in J_s} \sum_{i\in I} q_{ij} z_{ij} \Big), \tag{7}$$

$$x_i + \sum_{k \mid i \succcurlyeq_j k} z_{kj} \leqslant 1, \qquad i \in I, \quad j \in J, \tag{8}$$

$$x_i + z_i^s \le 1, \qquad i \in I, \quad s \in S, \tag{9}$$

$$z_i^s \geqslant z_{ij}, \qquad i \in I, \quad s \in S, \quad j \in J_s, \tag{10}$$

$$z_i^s, z_{ij} \in \{0,1\}, \qquad i \in I, \quad s \in S, \quad j \in J_s. \tag{11}$$

The objective function (1) expresses the amount of profit of the Leader if the worst (in relation to the decision made by the Leader) demand scenario is realized. The constraints (2) guarantee that the Leader serves the consumer only by the facilities which are more preferable for this consumer than any facility of the Follower. Moreover, from these constraints it follows that the share of the satisfied demand of each consumer cannot exceed one. The constraints (3) guarantee that the consumers are served only by the opened facilities. From the constraints (4) it follows that each facility satisfies the needs of consumers in the volume not exceeding the selected capacity of this facility.

The objective function (7) is the sum of the profits received by the Follower for all scenarios of demand. Note that maximizing this quantity is equivalent to maximizing the profit of the Follower separately for each scenario. The constraints (9) mean that the Follower cannot open a facility already opened by the Leader. The constraints (8) and (10) have the same meaning as the constraints (2) and (3).

In this bilevel model we denote the upper level problem (1)–(6) by $\mathcal{L}$, and the lower level problem (7)–(11), by $\mathcal{F}$. The entire model (1)–(11) will be denoted by $(\mathcal{L}, \mathcal{F})$.

## 2. AN OPTIMISTIC FEASIBLE SOLUTION

Let $X = ((x_i), (x_{ij}), (v_i))$ be a feasible solution of Problem $\mathcal{L}$ for given vectors $\tilde{z}^s = (\tilde{z}_i^s)$, $i \in I$, $s \in S$, and let $\widetilde{Z} = ((\tilde{z}_i^s), (\tilde{z}_{ij}))$ be the optimal solution of Problem $\mathcal{F}$ for a given vector $x = (x_i)$, $i \in I$. Then the pair $(X, \widetilde{Z})$ will be called a *feasible solution* of the bilevel problem $(\mathcal{L}, \mathcal{F})$. In what follows, we will consider the feasible solutions $(X, \widetilde{Z})$ with the optimal values of the variables $(x_{ij})$, i.e., providing maximum of the objective function of Problem $\mathcal{L}$ for the fixed vectors $x = (x_i)$ and $\tilde{z}^s = (\tilde{z}_i^s)$, $s \in S$.

Let us denote by $F(Z)$ the value of the objective function (7) of Problem $\mathcal{F}$ on the feasible solution $Z$, and by $L(X, \widetilde{Z})$, the value of the objective function (1) of Problem $(\mathcal{L}, \mathcal{F})$ on the feasible solution $(X, \widetilde{Z})$. A feasible solution $(X, \widetilde{Z})$, where $X = ((x_i), (x_{ij}), (v_i))$, will be called an *optimistic feasible solution* of Problem $(\mathcal{L}, \mathcal{F})$, provided $L(X, \widetilde{Z}) \geq L(X', \widetilde{Z}')$ for each feasible solution $(X', \widetilde{Z}')$, where $X' = ((x_i), (x'_{ij}), (v'_i))$. A feasible solution $(X^*, \widetilde{Z}^*)$ will be called an *optimistic optimal solution* or just an *optimal solution* of Problem $(\mathcal{L}, \mathcal{F})$ if $L(X^*, \widetilde{Z}^*) \geq L(X, \widetilde{Z})$ for every feasible solution $(X, \widetilde{Z})$.

Consider the problem of finding an optimistic optimal solution for the model $(\mathcal{L}, \mathcal{F})$. This problem can be written as follows:

$$\max_{(x_i),(x_{ij}),(v_i)} \max_{(\tilde{z}_i^s),(\tilde{z}_{ij})} \Big( -\sum_{i\in I}(f_i x_i + c_i v_i) + \min_s \sum_{i\in I}\sum_{j\in J_s} p_{ij} x_{ij} \Big)$$

under the conditions (2)–(11). We will use the same notation $(\mathcal{L}, \mathcal{F})$ for this problem as for the model (1)–(11).

For Problem $(\mathcal{L}, \mathcal{F})$ under study, let us consider the question of constructing an optimistic feasible solution in the case of a given $(0,1)$-vector $x = (x_i)$, $i \in I$. Such a solution can be obtained by the following two steps:

At the first step, Problem $\mathcal{F}$ is solved for a given vector $x$, and the optimal value $F^*$ of the objective function is calculated. At the second step, we solve the auxiliary problem

$$\max_{(x_{ij}),(v_i)} \max_{(z_i^s),(z_{ij})} \Big( -\sum_{i\in I} c_i v_i + \min_{s\in S} \sum_{i\in I}\sum_{j\in J_s} p_{ij} x_{ij} \Big) \tag{12}$$

under the conditions

$$z_i^s + \sum_{k \mid i \succ_j k} x_{kj} \leq 1, \qquad i \in I, \quad s \in S, \quad j \in J_s, \tag{13}$$

$$\sum_{s\in S} \Big( -\sum_{i\in I} g_i z_i^s + \sum_{i\in I}\sum_{j\in J_s} q_{ij} z_{ij} \Big) \geq F^* \tag{14}$$

and the constraints (3)–(5) and (8)–(11).

It is easy to see that the optimal solution of this problem gives an optimistic feasible solution for $(\mathcal{L}, \mathcal{F})$. We will refer to this solution as a *solution generated by the vector* $x$. It is also clear that the optimal solution of $(\mathcal{L}, \mathcal{F})$ is an optimistic feasible solution generated by some $(0,1)$-vector.

It follows that $(\mathcal{L}, \mathcal{F})$ can be represented as a maximization problem for some pseudo-Boolean function $f(x)$ of the variables $x_i$, $i \in I$. The value of this function on the vector $x = (x_i)$ is equal to the value of the objective function (12) of the problem $(\mathcal{L}, \mathcal{F})$ on the optimistic feasible solution generated by the vector $x$. To calculate the value of the pseudo-Boolean function $f(x)$, we need to solve Problem $\mathcal{F}$ and the auxiliary problem (12)–(14), (3)–(5), and (8)–(11).

## 3. UPPER BOUND

To construct exact algorithms for maximizing the pseudo-Boolean function $f(x)$ based on the methods of implicit enumeration, we need an efficient way of calculating the upper bound of the values of $f(x)$ on the subsets formed by $(0,1)$-vectors. For various schemes of implicit enumeration, it is convenient to use as such subsets the subsets defined by partial $(0,1)$-vectors.

A vector $y = (y_i)$, $i \in I$, whose entries take the values 0, 1, and $*$, will be called a *partial* $(0,1)$-*vector* or *partial solution*. For the partial solution $y$, we put

$$I^0(y) = \{i \in I \mid y_i = 0\}, \qquad I^1(y) = \{i \in I \mid y_i = 1\}.$$

The vector $x = (x_i)$, $i \in I$, is an *extension* of the partial solution $y$ if $x_i = y_i$ for all $i \in I^0(y) \cup I^1(y)$. We denote the set of all extensions of the partial solution $y$ by $P(y)$.

Consider the problem of computing the upper bound for $\max_{x \in P(y)} f(x)$ for an arbitrary partial solution $y$ that is not a $(0,1)$-vector.

The problem $\mathcal{L}$ with additional constraints $x_i = y_i$, $i \in I^0(y) \cup I^1(y)$, will be denoted by $\mathcal{L}(y)$. An upper bound for the values of the objective function of Problem $(\mathcal{L}(y), \mathcal{F})$ will be the sought-for upper bound for the values of the pseudo-Boolean function $f(x)$ for $x \in P(y)$.

To calculate an upper bound of the objective function of Problem $(\mathcal{L}(y), \mathcal{F})$, we construct an estimating mixed-integer programming problem (MIP) that is obtained from Problem $\mathcal{L}(y)$ by adding some constraints, which hold for optimistic feasible solutions, and excluding from it the restrictions containing the values of the variables of Problem $\mathcal{F}$. The technique for constructing these constraints is based on the estimating subsets $I_j(y)$, $j \in J$ [1, 2].

Let us construct the estimating subsets as applied to Problem $(\mathcal{L}(y), \mathcal{F})$ under study.

Given $j_0 \in J_s$, $s \in S$, we formulate the rules that allow us to determine whether $i_0 \in I$ belongs to $I_{j_0}(y)$.

First of all, we determine which facilities $i \in I$ of the Follower will not be open under realization of the scenario $s \in S$ if the Leader's facilities from $I' \subseteq I$ are open. To this end, we consider $J_s(I') = \{j \in J_s \mid i \succ_j \alpha_j(I')\}$. Then the Follower's facility $i \in I$ will not be open if it belongs to

$$M_s(I') = \Big\{i \in I \mid g_i > \sum_{j \in J_s(I')} q_{ij}\Big\}.$$

Let $N_{j_0}(i_0) = \{i \in I \mid i \succ_{j_0} i_0\}$. We assume that $i_0 \notin I_{j_0}(y)$ if $i_0 \in I^0(y)$ or $\alpha_{j_0}(I^1(y)) \succ_{j_0} i_0$. We put $i_0 \in I_{j_0}(y)$ if $N_{j_0}(i_0) = \varnothing$. Let none of these relations be satisfied and $M = M_s(I^1(y) \cup i_0) \cap I^0(y)$. Consider the set

$$J(i_0) = \{j \in J_s \ \mid \ \text{if } \ i \succ_j \alpha_j(I^1(y) \cup \{i_0\}) \ \text{ then } \ i \in N_{j_0}(i_0) \cup M\},$$

and also for each $k \in N_{j_0}(i_0)$ consider

$$J(k, i_0) = \big\{j \in J(i_0) \mid k \succ_j \alpha_j(I^1(y) \cup \{i_0\})\big\}, \qquad Q_k = \sum_{j \in J(k,i_0)} q_{kj}.$$

We say $i_0 \notin I_{j_0}(y)$ if there is a profitable facility $k \in N_{j_0}(i_0)$, i.e. such that $Q_k > g_k$. If there does not exist $k \in N_{j_0}(i_0)$ with the indicated property then $i_0 \in I_{j_0}(y)$.

**Lemma 1**. *Let $(X, \widetilde{Z})$, where $X = ((x_i), (x_{ij}), (v_i))$ and $\widetilde{Z} = \big((\tilde{z}_i^s), (\tilde{z}_{ij})\big)$, be an optimistic feasible solution of Problem $(\mathcal{L}(y), \mathcal{F})$ generated by $x \in P(y)$. Then if $\alpha_j(x) \notin I_j(y)$ for some $j \in J_s$, $s \in S$, then $z_i^s = 1$ for some $i \in N_j(\alpha_j(x))$.*

*Proof.* Suppose the contrary: Let $i = \alpha_j(x) \notin I_j(y)$ for some $j \in J_s$ and, in addition, $\tilde{z}_k^s = 0$ for each $k \in N_j(i)$. Note that $N_j(i) \neq \varnothing$ since $i \notin I_j(i)$. Let

$$M = M_s(I^1(y) \cup \{i\}) \cap I^0(y).$$

Then $x_k = \tilde{z}_k^s = 0$ for $k \in N_j(i) \cup M$. Since $i \notin I_j(y)$ and, in addition, $i \notin I^0(y)$ and $N_j(i) \cap I^1(y) = \varnothing$; therefore, by the construction of $I_j(y)$, there exist $k \in N_j(i)$ and the set $J(k,i)$ such that $Q_k > g_k$.

Let us construct a new solution $Z = \big((z_i^s), (z_{ij})\big)$ of Problem $\mathcal{F}$, that differs from $\widetilde{Z}$ only by the fact that $z_k^s = 1$ and $z_{kj} = 1$ for $j \in J(k,i)$. It is a feasible solution of $\mathcal{F}$. Moreover, for the values of the objective function of Problem $\mathcal{F}$ on the solutions $Z$ and $\widetilde{Z}$, we have $F(Z) - F(\widetilde{Z}) = Q_k - g_k > 0$. This contradicts the feasibility of the solution $(X, \widetilde{Z})$. Lemma 1 is proved. □

Lemma 1 directly yields

**Corollary 1.** *Let $(X, \widetilde{Z})$ be an optimistic feasible solution of Problem $(\mathcal{L}(y), \mathcal{F})$ generated by the vector $x \in P(y)$. If $\alpha_j(x) \notin I_j(y)$ for some $j \in J$ then*

$$\sum_{i \in I} x_{ij} = 0.$$

To write the established property of optimistic feasible solutions in the form of linear constraints, we introduce new nonnegative variables $t_{ij}$, $i \in I$, $j \in J$. The variable $t_{ij}$ takes the value 1 if $i = \alpha_j(x)$, and the value 0 otherwise.

**Corollary 2.** *Let $(X, \widetilde{Z})$ be optimistic feasible solution of Problem $(\mathcal{L}(y), \mathcal{F})$ generated by the vector $x \in P(y)$. Then the following hold:*

$$x_i \geq t_{ij}, \qquad i \in I, \quad j \in J, \tag{15}$$

$$\sum_{i \in I} t_{ij} = 1, \qquad j \in J, \tag{16}$$

$$x_i + \sum_{k \,|\, i \succ_j k} t_{kj} \leq 1, \qquad i \in I, \quad j \in J, \tag{17}$$

$$\sum_{k \in I} x_{kj} \leq 1 - t_{ij} + \sum_{k \in I_j(y)} t_{kj}, \qquad i \in I, \quad j \in J, \tag{18}$$

$$t_{ij} \geq 0, \qquad i \in I, \quad j \in J. \tag{19}$$

Indeed, according to the constraints (15)–(17), (19), the variables $t_{ij}$ take the value 1 if $i = \alpha_j(x)$, and 0 otherwise. If for some $i \in I$ and $j \in J$ we have $t_{ij} = 0$ then (18) is fulfilled. On the other hand, if $t_{ij} = 1$ and $i \notin I_j(y)$ then, by Lemma 1, the left-hand side of the inequality equals zero and, hence, the inequality holds.

Note that Lemma 1 also yields a more general relation:

**Corollary 3.** *Let $(X, \widetilde{Z})$ be an optimistic feasible solution of Problem $(\mathcal{L}(y), \mathcal{F})$ generated by the vector $x \in P(y)$. If $\alpha_j(x) \notin I_j(y)$ for some $j \in J_s$, $s \in S$, then*

$$\sum_{i \,|\, \omega_r(N_j(\alpha_j(x))) \succcurlyeq_r i} x_{ir} = 0, \qquad r \in J_s.$$

Indeed, by Lemma 1, $\tilde{z}_k^s = 1$ for some $k \in N_j(\alpha_j(x))$. Clearly, we have $k \succcurlyeq_r \omega_r(N_j(\alpha_j(x)))$ for all $r \in J_s$. Then, by (2), $x_{ir} = 0$ for every $i \in I$ such that $\omega_r(N_j(\alpha_j(x))) \succcurlyeq_r i$.

Using the introduced variables $t_{ij}$, $i \in I$ and $j \in J$, these relations can be written as linear constraints

$$\sum_{k \,|\, \omega_r(N_j(i)) \succcurlyeq_r k} x_{kr} \leq 1 - t_{ij} + \sum_{k \in I_j(y)} t_{kj}, \qquad i \in I, \quad s \in S, \quad j, r \in J_s. \tag{18$'$}$$

If $t_{ij} = 0$ for some $i \in I$ and $j \in J_s$, $s \in S$, then (18′) is fulfilled for all $r \in J_s$. Let $t_{ij} = 1$. In the case when $i \in I_j(y)$, the right-hand side equals 1 and the inequality is satisfied. Otherwise, the left-hand side equals 0 by Corollary 3.

Let us construct a system of additional estimating subsets $\{I_j^{hr}(y)\}$ for $s \in S$, $j, r \in J_s$, $j \neq r$, and $h \in I_r(y)$, allowing us to strength the constraints on the variables $x_{ij}$, $i \in I$ and $j \in J$, for an optimistic feasible solution $(X, \widetilde{Z})$ generated by the vector $x \in P(y)$, in the case when $\alpha_j(x) \in I_j(y)$.

Given $s \in S$, $j_0, r \in J_s$, $j_0 \neq r$, and $h \in I_r(y)$, we formulate the rules that allow us to determine for each $i \in I$ whether it belongs to the set $I_{j_0}^{hr}(y)$.

Let $i_0 \notin I_{j_0}^{hr}(y)$ if there holds at least one of the following four conditions:

$$i_0 \in I^0(y), \qquad i_0 \succ_r h, \qquad (I^1(y) \cup \{h\}) \cap N_{j_0}(i_0) \neq \varnothing, \qquad i_0 \notin I_{j_0}(y).$$

Suppose that none of these relations is satisfied; and put

$$N = N_{j_0}(i_0) \cup N_r(h), \qquad M = M_s(I^1 \cup \{i_0, h\}) \cap I^0(y).$$

We will assume that $i_0 \in I_{j_0}^{hr}(y)$ if $N = \varnothing$. Let $N \neq \varnothing$. Consider the set

$$J(i_0, h) = \{j \in J_s \mid \text{ if } i \succ_j \alpha_j(I^1(y) \cup \{i_0, h\}) \text{ then } i \in N \cup M\}.$$

Given $k \in N$, we also consider

$$J(k, i_0, h) = \{j \in J(i_0, h) \mid k \succ_j \alpha_j(I^1(y) \cup \{i_0, h\})\}, \qquad Q_k = \sum_{j \in J(k, i_0, h)} q_{kj}.$$

Let $i_0 \notin I_{j_0}^{hr}(y)$ if there is a profitable facility $k \in N$, i.e. such that $Q_k > g_k$. If there does not exist $k \in N$ with the indicated property then we put $i_0 \in I_{j_0}^{hr}(y)$.

The basic property of the set $I_j^{hr}(y)$ is established in

**Lemma 2.** *Let $(X, \widetilde{Z})$, $X = ((x_i), (x_{ij}), (v_i))$, $\widetilde{Z} = \big((\tilde{z}_i^s), (\tilde{z}_{ij})\big)$ be an optimistic feasible solution of Problem $(\mathcal{L}(y), \mathcal{F})$ generated by $x \in P(y)$. Then if $h = \alpha_r(x) \in I_r(x)$ for some $r \in J_s$, $s \in S$, and $\alpha_j(x) \notin I_j^{hr}(y)$ for some $j \in J_s$, $j \neq r$, then $\tilde{z}_i^s = 1$ for some $i \in N_j(\alpha_j(x)) \cup N_r(\alpha_r(x))$.*

*Proof.* Let $j, r \in J_s$, $s \in S$, and let $h = \alpha_r(x) \in I_r(y)$, $i = \alpha_j(x) \notin I_j^{hr}(y)$. Let us note, first of all, that if $i \notin I_j(y)$ then, by Lemma 1, we have $\tilde{z}_k^s = 1$ for some $k \in N_j(i)$. Therefore, we will assume that $i \in I_j(y)$. Note also that since $i \notin I_j^{hr}(y)$; therefore, by the construction of this set, we have $N = N_j(i) \cup N_r(h) \neq \varnothing$. Let us assume the contrary, and let $\tilde{z}_k^s = 0$ for all $k \in N$. Then for the considered optimistic feasible solution $(X, \widetilde{Z})$ we have $x_k = \tilde{z}_k^s = 0$ for all $k \in N \cup M$. Since $i \notin I_j^{hr}(y)$ and, in addition,

$$i \notin I^0(y), \qquad h \not\succ_r i, \qquad (I^1(y) \cup \{h\}) \cap N_j(i) = \varnothing, \qquad i \in I_j(y);$$

by the construction of $I_j^{hr}(y)$, there exist $k \in N$ and a set $J(k, i, h) \subseteq J(i, h)$ such that

$$Q_k = \sum_{j \in J(k, i, h)} q_{kj} > g_k.$$

Let us construct a new solution $Z = \big((z_i^s), (z_{ij})\big)$ of Problem $\mathcal{F}$ that differs from the solution $\widetilde{Z}$ only by the fact that $z_k^s = 1$ and $z_{kj} = 1$ for $j \in J(k, i, h)$. It is a feasible solution of Problem $\mathcal{F}$ for which

$$F(Z) - F(\widetilde{Z}) = Q_k - g_k > 0.$$

We arrive at a contradiction with the feasibility of $(X, \widetilde{Z})$. Lemma 2 is proved. ◻

**Corollary 4.** *Let* $(X, \widetilde{Z})$, $X = ((x_i), (x_{ij}), (v_i))$, *be an optimistic feasible solution of Problem* $(\mathcal{L}(y), \mathcal{F})$, *generated by* $x \in P(y)$. *Then if* $h = \alpha_r(x) \in I_r(y)$ *and* $\alpha_j(x) \notin I_j^{hr}(y)$ *for some* $j, r \in J_s$, $s \in S$ *and* $j \neq r$, *then*

$$\sum_{i\in I} x_{ij} + \sum_{i\in I} x_{ir} \leq 1.$$

Using the previous variables $t_{ij}$, $i \in I$, $j \in J$, this statement is written as follows:

$$\sum_{i\in I} x_{ij} + \sum_{i\in I} x_{ir} \leq 2 - t_{hr} + \sum_{i\in I_j^{hr}(y)} t_{ij}, \qquad j, r \in J_s, \quad s \in S, \quad h \in I_r(y). \tag{20}$$

Indeed, if $t_{hr} = 1$ and $t_{ij} = 0$ for every $i \in I_j^{hr}$ then the conditions of Lemma 2 hold and at least one of the sums $\sum_{i\in I} x_{ij}$ or $\sum_{i\in I} x_{ir}$ equals zero.

The estimating problem obtained from $\mathcal{L}(y)$ by including the additional restrictions (15)–(17), (18′), (19), and (20) and excluding those connecting it with Problem $\mathcal{F}$, can be written as follows:

$$\max_{(x_i),(x_{ij}),(v_i)} \Big( -\sum_{i\in I} (f_i x_i + c_i v_i) + \min_{s\in S} \sum_{j\in J_s} \sum_{i\in I} p_{ij} x_{ij} \Big), \tag{21}$$

$$x_i \geqslant x_{ij}, \qquad i \in I, \quad j \in J, \tag{22}$$

$$\sum_{j\in J_s} a_{ij} x_{ij} \leq v_i, \qquad i \in I, \quad s \in S, \tag{23}$$

$$x_i \geq t_{ij}, \qquad i \in I, \quad j \in J, \tag{24}$$

$$\sum_{i\in I} t_{ij} = 1, \qquad j \in J, \tag{25}$$

$$x_i + \sum_{k \,|\, i \succ_j k} t_{kj} \leq 1, \qquad i \in I, \quad j \in J, \tag{26}$$

$$\sum_{k \,|\, \omega_r(N_j(i)) \succcurlyeq_r k} x_{kr} \leq 1 - t_{ij} + \sum_{k\in I_j(y)} t_{kj}, \qquad i \in I, \quad s \in S, \quad j, r \in J_s, \tag{27}$$

$$\sum_{i\in I} x_{ij} + \sum_{i\in I} x_{ir} \leq 2 - t_{hr} + \sum_{i\in I_j^{hr}(y)} t_{ij}, \qquad j, r \in J_s, \quad s \in S, \quad h \in I_r(y), \tag{28}$$

$$x_i = y_i, \qquad i \in I^0(y) \cup I^1(y), \tag{29}$$

$$x_i \in \{0, 1\}, \qquad t_{ij}, x_{ij} \in [0, 1], \qquad v_i \geq 0, \qquad i \in I, \quad j \in J. \tag{30}$$

Denote this problem by $\mathcal{B}(y)$, and by $B(X, T)$, the value of its objective function at the feasible solution $(X, T)$, where $T = (t_{ij})$.

**Theorem.** *Let* $(X^*, T^*)$ *be an optimal solution of Problem* $\mathcal{B}(y)$. *Then*

$$\max_{x\in P(y)} f(x) \leq B(X^*, T^*).$$

*Proof.* Let $x \in P(y)$ and let $(X, \widetilde{Z})$ be an optimistic solution of Problem $(\mathcal{L}(y), \mathcal{F})$ generated by the vector $x$. Given $i \in I$ and $j \in J$, we put $t_{ij} = 1$ if $i = \alpha_j(x)$, and $t_{ij} = 0$ otherwise. Note that $(X, T)$, where $T = (t_{ij})$, is a feasible solution of Problem $\mathcal{B}(y)$. Indeed, the constraints (22), (23), and (29) are satisfied since $X$ is a feasible solution of Problem $\mathcal{L}(y)$; the constraints (24)–(26) hold because of the choice of the values $t_{ij}$, $i \in I$ and $j \in J$; whereas the inequalities (27) and (28) are valid by Lemmas 1 and 2. Since the values of the objective functions of Problems $(\mathcal{L}(y), \mathcal{F})$ and $\mathcal{B}(y)$ on the solutions $(X, \widetilde{Z})$ and $(X, T)$ are equal, we have $f(x) = L(X, \widetilde{Z}) = B(X, T) \leq B(X^*, T^*)$.

The Theorem is proved. □

Note that Problem $\mathcal{B}(y)$ is an integer linear programming problem; therefore, from what is proved, we see that the calculation of the upper bound of the values of the pseudo-Boolean function $f(x)$ on the set $P(y)$ is reduced to the solution of an estimating problem in the form of MIP.

## CONCLUSION

In the paper, we study a new competitive facility location model constructed on the basis of the idea of the Stackelberg game. The model considers a situation of several alternative demand scenarios which differ both in the composition of consumers and in their preferences. The demand scenario becomes known after the placement of the Leader's facilities; therefore the Leader, when making a decision, is guided by the criterion of maximum profit in the worst case. In the model under consideration, it is also assumed that the costs of opening a facility depend on its capacity. Therefore, the Leader, deciding on the location of facilities, should determine their capacity and take into account various possible demand scenarios.

For the model under consideration, we formulate a problem of finding an optimistic optimal solution. We demonstrate that, for a given $(0,1)$-vector of locations of the Leader's facilities, it can be represented as a single-level problem. Thereby, the problem can be considered as a problem to maximize a pseudo-Boolean function. This allows us to construct some algorithms for finding the approximate solutions by use of various local enumeration schemes.

As applied to the problem under study, we modified the method for construction of the estimating problems for calculating the upper bounds of the values of the constructed pseudo-Boolean function on the subsets of solutions given by partial $(0,1)$-vectors. A new system of estimating subsets is proposed that allows us to supplement the estimating problems with new constraints, which increases both the accuracy of the obtained upper bounds and, consequently, the efficiency of the implicit enumeration algorithms by using these upper bounds.

The next step of our research includes the construction of algorithms for computing the exact and approximate solutions of the problem of competitive location of facilities using the developed method for constructing the upper bounds of the objective function values.

## ACKNOWLEDGMENTS

The authors were supported by the Russian Foundation for Basic Research (project no. 15–01–01446).

## REFERENCES

1. V. L. Beresnev, "On the Competitive Facility Location Problem with a Free Choice of Suppliers," Avtom. Telemekh. No. 4, 94–105 (2014) [Autom. Remote Control **75** (4), 668–676 (2014)].
2. V. L. Beresnev and A. A. Mel'nikov, "A Capacitated Competitive Facility Location Problem," Diskretn. Anal. Issled. Oper. **23** (1), 35–50 (2016) [J. Appl. Indust. Math. **10** (1), 61–68 (2016)].
3. A. A. Mel'nikov, "Randomized Local Search for the Discrete Competitive Facility Location Problem," Avtom. Telemekh. No. 4, 134–152 (2014) [Autom. Remote Control **75** (4), 700–714 (2014)].
4. M. G. Ashtiani, "Competitive Location: A State-of-Art Survey," Int. J. Ind. Eng. Comput. **7** (1), 1–18 (2016).
5. V. L. Beresnev, "Branch-and-Bound Algorithm for a Competitive Facility Location Problem," Comput. Oper. Res. **40** (8), 2062–2070 (2013).
6. I. A. Davydov, Yu. A. Kochetov, and E. Carrizosa, "A Local Search Heuristic for the $(r|p)$-Centroid Problem in the Plane," Comput. Oper. Res. **52**, Pt. B, 334–340 (2014).
7. S. Dempe, *Foundations of Bilevel Programming* (Kluwer Acad. Publ., Dordrecht, 2002).
8. T. Drezner, Z. Drezner, and P. Kalczynski, "A Leader–Follower Model for Discrete Competitive Facility Location," Comput. Oper. Res. **64**, 51–59 (2015).
9. H. A. Eiselt and G. Laporte, "Sequential Location Problems," European J. Oper. Res. **96** (2), 217–231 (1996).
10. A. Jakubovskis, "Strategic Facility Location, Capacity Acquisition, and Technology Choice Decisions under Demand Uncertainty: Robust vs. Nonrobust Optimization Approaches," European J. Oper. Res. **260** (3), 1095–1104 (2017).
11. A. Karakitsiou, *Modeling Discrete Competitive Facility Location* (Springer, Cham, 2015).
12. H. von Stackelberg, *The Theory of the Market Economy* (Oxford Univ. Press, Oxford, 1952).
13. Y. Zhang, L. V. Snyder, T. K. Ralphs, and Z. Xue, "The Competitive Facility Location Problem under Disruption Risks," Transp. Res., Part E: Logistics and Transportation Review **93**, 453–473 (2016).