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A Mathematical Model of Market Competition

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Abstract—We consider a mathematical model of decision making by a company attempting to win a market share. We assume that the company releases its products to the market under the competitive conditions that another company is making similar products. Both companies can vary the kinds of their products on the market as well as the prices in accordance with consumer preferences. Each company aims to maximize its profit. A mathematical statement of the decision-making problem for the market players is a bilevel mathematical programming problem that reduces to a competitive facility location problem. As regards the latter, we propose a method for finding an upper bound for the optimal value of the objective function and an algorithm for constructing an approximate solution. The algorithm amounts to local ascent search in a neighborhood of a particular form, which starts with an initial approximate solution obtained simultaneously with an upper bound. We give a computational example of the problem under study which demonstrates the output of the algorithm.

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INTRODUCTION

The monograph [1] on the market economy represents market competition by means of a multi-stage decision-making model. This model, also called the Stackelberg game, describes a situation of market competition in which, at the first stage, the leader company makes a decision and, at the second stage, the follower company takes that into account while deciding. Both companies pursue their own goals that are different in general.

In this article we propose a mathematical model of competition on a market for products of several kinds made by the two manufacturers, the leader company and the follower company, as they enter the market successively, under the assumption that the kinds of products and all prices are discrete. This model represents decision making by the market players as a multi-stage process. At the first stage, the leader company decides which products to offer and sets their prices. At the second stage, taking the decision of the leader company into account, the follower company makes its choice trying to offer more appealing products, possibly at lower prices, in order to capture some consumers. Finally, at the third stage, every consumer, basing on his own goals or preferences, buys some products and brings profit to either company. The goal of the leader company is to make a decision that yields the greatest profit on assuming that the follower company also tries to maximize its profit and captures some consumers. In the economics literature (see [2]) similar models are more often considered in the continuous formulation and studied primarily with the goal of finding the qualitative properties of solutions which enable the market players to reach an equilibrium.

We state the decision-making problem for the leader company as a bilevel integer programming problem that amounts to the well-known competitive facility location problem. The literature on this problem and its particular cases may presently be regarded as extensive [3–10]. Among these

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publications we should mention the article [5] which deals with the facility location with fixed cost. At the same time, few articles go beyond discussing the statements and propose methods for constructing optimal solutions or upper bounds for the values of the objective functions. We should note the articles [3, 9] which propose some algorithms for solving the problem under the assumption that the number of objects opened by the follower company is small or even equal to 1.

A method is proposed in [11] for computing an upper bound for the "linear" version of the competitive facility location problem. We extend the method to the general case of an competitive facility location problem. The algorithm for finding an upper bound reduces to solving the classical facility location problem. Also we propose an algorithm for constructing an approximate solution to the problem under study. The algorithm amounts to local ascent search in a neighborhood of a particular form. The output of this procedure is an approximate solution in the form of a local maximum. The procedure starts searching from an initial approximate solution obtained simultaneously with the upper bound.

Section 1 presents the mathematical model of market competition in the framework of which we formulate the decision-making problem for the leader company. This problem is stated as a well-known bilevel mathematical programming problem, i.e., the competitive facility location problem. Section 2 is devoted to studying the competitive facility location problem and constructing an algorithm for calculating an upper bound for the optimal value of the objective function. An algorithm for finding an approximate solution to the problem is described in Section 3. Section 4 contains a computational example of the proposed model which illustrates the algorithms for calculating an upper bound and constructing an approximate solution to the problem under study.

1. THE MATHEMATICAL MODEL

Let us construct a mathematical model that reflects the competition on the market of several products offered by the two companies, the leader and the follower. In this competition all market players make decisions: the leader company, the follower company, and the consumers of the products in question. It is convenient to express this decision making as the following three-stage process:

At the first stage, the leader company decides which products to offer and sets their prices. At the second stage, taking the decision of the leader company into account, the follower company makes its decision on the kinds of products to offer on the market and their prices. Finally, at the third stage, every consumer has the possibility to satisfy his demand by the products of both companies, buys some products and brings profit to either company, or satisfies his demand on a different market.

The decision-making problem for the leader company amounts to choosing a decision that yields the greatest profit on assuming that the follower company also tries to maximize its profit and captures some consumers. We assume further that the leader company, when making its decision, has a precise forecast concerning the kinds of products the follower company can offer and the range of possible prices for each kind. Moreover, we assume that both companies know the preferences of all consumers which take the prices into account.

Stating this problem formally, we use the following additional assumptions: assume that the profit of a company for each type of product is formed from the revenue minus the expenses from selling this type of product minus the so-called fixed expenses related, for instance, to logistics, advertisement, and so on. The revenue for the product of a given kind is determined by the number of the items sold and their price. Assume that the price range for the products of either company is not too wide; in fact, restricted to a few values, for instance, a low, an average, and a high price.

Let $I = \{1, \ldots, m\}$ denote the set of kinds of product on this market. Each $i \in I$ corresponds to a certain kind of product sold on the market at a particular price. We refer to this kind of product as type i. Therefore, if two types are products of the same physical kind but have different prices then different elements represent these types in I.

Let I_L and I_F denote the subsets of I consisting of the types offered by the leader company and the follower company respectively. Assume that $I_L \cup I_F = I$, but in general $I_L \cap I_F \neq \varnothing$.

Suppose that the following quantities are known for all $i \in I$:

- the price c_i at which type i products are sold;
- the specific expenses a_i related to making and selling type i products;
- the fixed cost f_i of the leader company related to making and selling type i products (assume that $f_i = \infty$ for $i \notin I_L$);

• the fixed cost g_i of the follower company related to making and selling type i products (assume that $g_i = \infty$ for $i \notin I_F$).

Let $J = \{1, \dots, n\}$ denote the set of consumers. Each element $j \in J$ stands for some consumer called consumer j. Assume that every consumer chooses products on the market to satisfy his demand, based on his own preferences. These preferences are determined by a linear order $\succ_j$ on I. The relation $i \succ_j k$ for $i, k \in I$ means that if type i and k products are present then consumer j prefers type i products. Assume also that, for all $i, k \in I$, the relation $i \succcurlyeq_j k$ means that either $i \succ_j k$ or $i = k$.

Given $i \in I$ and $j \in J$, let q_{ij} denote the number of units of the type i product that consumer j needs to satisfy his demand. Assume that $q_{ij} = 0$ if the type i product is not fitting for consumer j. For $i \in I$ and $j \in J$, let p_{ij} denote the quantity $q_{ij}(c_i - a_i)$ equal to the profit of the leader company or the follower company if consumer j chooses the type i product. For every $j \in J$, let A_j denote the set $\{i \in I \mid p_{ij} > 0\}$ of types of products admissible for consumer j. Assume that the market under consideration satisfies the demand of consumer j if at least one type of product in A_j is available.

In order to state the problem formally, introduce the following variables:

x_i shows whether the leader company offers products of type $i \in I$; here $x_i = 1$ if the answer is affirmative, and $x_i = 0$ otherwise;

x_{ij} shows whether the product of type $i \in I$ offered by the leader company is the most preferable product for consumer $j \in J$ among all types offered by the leader company; here $x_{ij} = 1$ if the answer is affirmative, and $x_{ij} = 0$ otherwise;

z_i shows whether the follower company offers products of type $i \in I$; here $z_i = 1$ if the answer is affirmative, and $z_i = 0$ otherwise;

z_{ij} shows whether the product of type $i \in I$ offered by the follower company is the most preferable product for consumer $j \in J$ among all types offered by both companies; here $z_{ij} = 1$ if the answer is affirmative, and $z_{ij} = 0$ otherwise.

Using this notation and variables, we can state the decision-making problem for the leader company as

$$\max_{(x_i),(x_{ij})} \Big\{ -\sum_{i\in I} f_i x_i + \sum_{j\in J} \Big(\sum_{i\in I} p_{ij} x_{ij}\Big)\Big(1 - \sum_{i\in I} \tilde{z}_{ij}\Big)\Big\}, \tag{1}$$

$$x_i + \sum_{k \mid i \succ_j k} x_{kj} \le 1, \qquad i \in I,\ j \in J, \tag{2}$$

$$x_i \ge x_{ij}, \qquad i \in I,\ j \in J, \tag{3}$$

$$x_i, x_{ij} \in \{0, 1\}, \qquad i \in I,\ j \in J, \tag{4}$$

$$((\tilde{z}_i), (\tilde{z}_{ij})) \text{ is the optimal solution to (6)–(8)}, \tag{5}$$

$$\max_{(z_i),(z_{ij})} \Big\{ -\sum_{i\in I} g_i z_i + \sum_{j\in J}\sum_{i\in I} p_{ij} z_{ij}\Big\}, \tag{6}$$

$$x_i + z_i + \sum_{k \mid i \succ_j k} z_{kj} \le 1, \qquad i \in I,\ j \in J, \tag{7}$$

$$z_i \ge z_{ij}, \qquad i \in I,\ j \in J, \tag{8}$$

$$z_i, z_{ij} \in \{0, 1\}, \qquad i \in I,\ j \in J. \tag{9}$$

The objective function (1) of this problem expresses the profit of the leader company taking into account the loss of revenue due to the follower company capturing part of consumers. The inequality (2) guarantees the fulfilment of the rule for consumer j choosing products to satisfy his demand. The same inequality guarantees that consumer j may choose at most one type of product. The restriction (3) means that consumer j may choose the products by the leader company only of type i present on the market. The objective function (6) and the restrictions (7) and (8) have similar meanings. The objective function (6) expresses the total profit made by the follower company, and (7) guarantees the fulfilment of the rule that consumer j chooses the best type of product among all types offered by both companies. Aside from that, (7) shows that products of the same type cannot be offered by both companies.

The mathematical statement (1)–(9) of the decision-making problem for the leader company in market competition amounts to the well-known competitive facility location problem [11]. This model is a bilevel integer programming problem [12]. Like every bilevel programming problem, it includes the upper level problem (1)–(4) called *problem* L, and the lower level problem (6)–(9) called *problem* F.

Let $X = ((x_i), (x_{ij}))$ denote an admissible solution to problem L. Given a fixed admissible solution X, denote the optimal solution to problem F by $\widetilde{Z} = ((\tilde{z}_i), (\tilde{z}_{ij}))$; and the set of optimal solutions to problem F, by $O(X)$. Let $L(X, \widetilde{Z})$ be the value of the objective function of problem L at an admissible solution X and an optimal solution $\widetilde{Z} \in O(X)$.

If we assume that, given an admissible solution X, an optimal solution $\widetilde{Z}$ is uniquely determined, i.e., $O(X)$ is a singleton, then $L(X, \widetilde{Y})$ is found uniquely for every admissible solution X. In this sense the problem (1)–(9) is well-posed, and the question never arises whether that is the optimal solution to problem L. An admissible solution X^* to problem L is an optimal solution to problem L whenever, given an admissible solution X, we have

$$L(X^*, \widetilde{Y}^*) \geq L(X, \widetilde{Y}), \qquad \widetilde{Y}^* \in O(X^*), \qquad \widetilde{Y} \in O(X).$$

However, if, for some admissible solutions X, the set $O(X)$ contains more than one element, and distinct $\widetilde{Z}_1 \in O(X)$ and $\widetilde{Z}_2 \in O(X)$ satisfy $L(X, \widetilde{Z}_1) \neq L(X, \widetilde{Z}_2)$, then the formulation (1)–(9) of the decision-making problem for the leader company is not well-posed. In order to make it well-posed, we must refine and reflect in the model the rule for choosing an optimal solution $\widetilde{Z} \in O(X)$ by the follower company. Assume that in this situation of market competition the follower company behaves *noncooperatively*: from all possible optimal solutions $\widetilde{Z} \in O(X)$ it chooses the solution which yields the smallest value $L(X, \widetilde{Z})$ of the objective function of problem L.

In the case of the noncooperative behavior of the follower company, we obtain a formulation of the decision-making problem for the leader company differing from (1)–(9) only in the condition (1) which becomes

$$\max_{(x_i),(x_{ij})} \min_{(\tilde{z}_i),(\tilde{z}_{ij})} \left\{ -\sum_{i\in I} f_i x_i + \sum_{j\in J} \left(\sum_{i\in I} p_{ij} x_{ij} \right) \left(1 - \sum_{i\in I} \tilde{z}_{ij} \right) \right\}. \tag{1'}$$

The problem (1′), (2)–(9) is a maximin bilevel integer programming problem. In this case, the optimal solution to the upper level problem L' is determined as follows:

Refer as an *admissible solution* to (1′), (2)–(9) to the pair $(X, \overline{Z})$, where X is an admissible solution to problem L' and $\overline{Z} \in O(X)$ is an optimal solution satisfying

$$L(X, \overline{Z}) = \min_{\widetilde{Z} \in O(X)} L(X, \widetilde{Z}).$$

An admissible solution $(X^*, \overline{Z}^*)$ to (1′), (2)–(9) is called *optimal* whenever $L(X^*, \overline{Z}^*) \geq L(X, \overline{Z})$ for every admissible solution $(X, \overline{Z})$. Then an admissible solution X^* to problem L' is an optimal solution to this problem if we can find $\overline{Z}^* \in O(X^*)$ such that the pair $(X^*, \overline{Z}^*)$ is an optimal solution to (1′), (2)–(9).

An admissible solution X to problem L' is called an *approximate solution* to problem L' whenever we can find $\overline{Z} \in O(X)$ such that the pair $(X, \overline{Z})$ is an admissible solution to (1′), (2)–(9). It is clear that every admissible solution X can serve as an approximate solution to problem L'. The corresponding optimal solution $\overline{Z} \in O(X)$ to problem F is determined using an algorithm consisting of the two stages:

At stage 1, for a fixed solution X, we solve problem F and calculate some optimal solution $\widetilde{Z} = ((\tilde{z}_i), (\tilde{z}_{ij}))$ to problem F.

At stage 2, for fixed solutions X and $\widetilde{Z}$, we solve the auxiliary problem

$$\min_{(z_i),(z_{ij})} \sum_{j\in J}\Big(\sum_{i\in I} p_{ij}x_{ij}\Big)\Big(1-\sum_{i\in I} z_{ij}\Big), \tag{10}$$

$$x_i + z_i + \sum_{k|i\succ_j k} z_{kj} \le 1, \qquad i\in I,\ j\in J, \tag{11}$$

$$z_i \ge z_{ij}, \quad i\in I,\ j\in J, \tag{12}$$

$$-\sum_{i\in I} g_i z_i + \sum_{j\in J}\sum_{i\in I} p_{ij}z_{ij} \ge -\sum_{i\in I} g_i\tilde{z}_i + \sum_{j\in J}\sum_{i\in I} p_{ij}\tilde{z}_{ij}, \tag{13}$$

$$z_i, z_{ij} \in \{0,1\}, \qquad i\in I,\ j\in J. \tag{14}$$

An optimal solution $\overline{Z} = ((\bar{z}_i)(\bar{z}_{ij}))$ to this problem is a required optimal solution to problem F.

2. AN UPPER BOUND FOR THE OPTIMAL VALUE OF THE OBJECTIVE FUNCTION OF (1′), (2)–(9)

Let us construct an approximate solution X to problem L' for which $L(X,\overline{Z})$ is close to the optimal value of the objective function of (1′), (2)–(9). In order to estimate this proximity, calculate an upper bound for the optimal value of the objective function of (1′), (2)–(9).

For every $j\in J$, put

$$p_j = \max_{i\in A_j\cap I_L} p_{ij}.$$

A trivial upper bound for $L(X,\overline{Z})$ for every admissible solution $(X,\overline{Z})$ with $X=((x_i)(x_{ij}))$ and $\overline{Z}=((\bar{z}_i)(\bar{z}_{ij}))$ follows from

$$\Big(\sum_{i\in I} p_{ij}x_{ij}\Big)\Big(1-\sum_{i\in I}\bar{z}_{ij}\Big) \le p_j \max_{i\in A_j\cap I_L} x_i$$

for every $j\in J$. This inequality holds since if $x_i=0$ for every $i\in A_j\cap I_L$ then $\sum_{i\in I} p_{ij}x_{ij}=0$.

For every $j\in J$, define a set $I_j\subset A_j\cap I_L$ giving a similar but stronger inequality, which enables us to obtain a sharper upper bound.

Take $i\in A_j\cap I_L$ and consider the set

$$N(i)=\{k\in A_j \mid k\succ_j i\},$$
$$J(i)=\{s\in J \mid i\succcurlyeq_s k \text{ for every } k\in A_s\setminus N(i)\}.$$

Observe that $J(i)\ne\varnothing$ since $j\in J(i)$.

If $N(i)=\varnothing$ then assume that $i\in I_j$ by definition. Suppose that $N(i)\ne\varnothing$. Given $k\in N(i)\cap I_F$, construct the set

$$J(k,i)=\{s\in J(i)\mid k\succ_s i\}.$$

Assume that $i\in I_j$ if

$$g_k > \sum_{s\in J(k,i)} p_{ks} - \sum_{s\notin J(i)} \max\{0;\max_{l\in I_F\setminus N(i)|k\succ_s l\succ_s i}(p_{ls}-p_{ks})\}$$

for every $k\in N(i)\cap I_F$. Observe that the set I_j can be empty since, even for $i_0\in A_j\cap I_L$ such that $i_0\succcurlyeq_j i$ for every $i\in A_j\cap I_L$, the set $N(i_0)$ can be nonempty, and there is $k\in N(i_0)\cap I_F$ for which the inequality is violated.

The significance of I_j is clear from the following lemma, which establishes that if the leader company plans to satisfy consumer j by its products, but fails to offer the type i products for all $i\in I_j$, then consumer j will be captured by the follower company. In particular, if I_j is empty then consumer j will be captured by the follower company independently of the offers by the leader company.

Lemma 1. *For every admissible solution $(X,\overline{Z})$, $X=((x_i)(x_{ij}))$, $\overline{Z}=((\bar{z}_i)(\bar{z}_{ij}))$, to (1′), (2)–(9) and every $j\in J$ such that $\sum_{i\in I}p_{ij}x_{ij}>0$ and $x_i=0$ for every $i\in I_j$, we have $\sum_{i\in I}\bar{z}_{ij}=1$.*

Proof. Suppose that the required equality fails for some $j \in J$ and that $x_{i_0 j} = 1$ for some $i_0 \in I$. Since $p_{i_0 j} > 0$, it follows that $i_0 \in A_j$. Consider the set $N(i_0)$ and observe that $\bar{z}_k = 0$ for every $k \in N(i_0)$. Consider also the set $J(i_0)$ and observe that, for every $s \in J(i_0)$, the relation $i_0 \succ_s i$ holds for every $i \in A_s$ such that $\bar{z}_i = 1$. Since $x_{i_0} = 1$, it follows that $i_0 \notin I_j$, and consequently, there is $k \in N(i_0) \cap I_F$ for which there exists $J(k, i_0) \subset J(i_0)$ such that

$$g_k \le \sum_{s \in J(k,i_0)} p_{ks} - \sum_{s \in J(i_0)} \max\{0; \max_{l \in I_F \setminus N(i_0) | k \succ_s l \succ_s i_0} (p_{ls} - p_{ks})\}.$$

Moreover, for every $s \in J(k, i_0)$, we have $k \succ_s i_0$ and $i_0 \succ_s i$ for every $i \in A_s$ such that $\bar{z}_i = 1$. Put $I_0 = \{i \in I \mid x_i + \bar{z}_i = 1\}$. For every $s \notin J(i_0)$, let $i(s)$ be an element $k_0 \in I_0$ such that $k_0 \succcurlyeq_s i$ for every $i \in I_0$. Put

$$S = \{s \notin J(i_0) \mid k \succ_s i(s)\}, \qquad S_F = \{s \in S \mid \bar{z}_{i(s)} = 1\}.$$

Consider the solution $\bar{z}' = \big((\bar{z}'_i), (\bar{z}'_{is})\big)$ differing from the original solution $\bar{z}$ in that

$$\bar{z}'_k = 1, \qquad \bar{z}'_{ks} = 1, \quad s \in J(k, i_0), \qquad \bar{z}'_{ks} = 1, \quad s \in S, \qquad \bar{z}'_{i(s)s} = 1, \quad s \in S_F.$$

The difference between the values of the objective function (6) at $\bar{z}'$ and the original optimal solution $\bar{z}$ satisfies

$$\begin{aligned} -g_k + \sum_{s \in J(k,i_0)} p_{ks} + \sum_{s \in S} p_{ks} - \sum_{s \in S_F} p_{i(s)s} &\ge -g_k + \sum_{s \in J(k,i_0)} p_{ks} - \sum_{s \in S_F} (p_{i(s)s} - p_{ks}) \\ &\ge -g_k + \sum_{s \in J(k,i_0)} p_{ks} - \sum_{s \notin J(i_0)} \max\{0; \max_{l \in I_F \setminus N(i_0) | k \succ_s l \succ_s i_0} (p_{ls} - p_{ks})\} \ge 0. \end{aligned}$$

This means that $\bar{z}'$ is also an optimal solution. Moreover, $L(X, \overline{Z}') < L(X, \overline{Z})$. This contradicts the assumption that $(X, \overline{Z})$ is an admissible solution. The proof of the lemma is complete. □

Given $j \in J$, put $p_j = \max_{i \in I_j} p_{ij}$. If $I_j = \varnothing$ then $p_j = 0$.

Lemma 2. *For every admissible solution* $(X, \overline{Z})$, $X = ((x_i)(x_{ij}))$, $\overline{Z} = ((\bar{z}_i)(\bar{z}_{ij}))$, *to* (1′), (2)–(9) *and every* $j \in J$, *we have*

$$\Big(\sum_{i \in I} p_{ij} x_{ij}\Big)\Big(1 - \sum_{i \in I} \bar{z}_{ij}\Big) \le p_j \max_{i \in I_j} x_i.$$

Proof. If $\sum_{i \in I} p_{ij} x_{ij} = 0$ then the inequality holds. If $p_{ij} x_{ij} > 0$ for some $i \in I_j$ then the inequality holds as well. Suppose that $\sum_{i \in I} p_{ij} x_{ij} > 0$ and $x_i = 0$ for all $i \in I_j$. Then Lemma 1 yields $\sum_{i \in I} \bar{z}_{ij} = 1$ and the required inequality holds. The proof of the lemma is over. □

Define the matrix (h_{ij}), $i \in I$, $j \in J$, by putting

$$h_{ij} = \begin{cases} 0, & \text{if } i \in I_j, \\ 1 & \text{otherwise.} \end{cases}$$

We have the following

Theorem 1. *The quantity*

$$\sum_{j \in J} p_j - \min_{(x_i)} \Big\{ \sum_{i \in I} f_i x_i + \sum_{j \in J} \min_{i | x_i = 1} p_j h_{ij} \Big\}$$

is an upper bound for the optimal value of the objective function of (1′), (2)–(9).

Proof. Given an admissible solution $(X,\overline{Z})$, $X=((x_i)(x_{ij}))$, $\overline{Z}=((\bar{z}_i)(\bar{z}_{ij}))$, to (1′), (2)–(9), the previous lemma and

$$\max_{i\in I_j} x_i = \max_{i|x_i=1}(1-h_{ij})$$

yield

$$\begin{aligned} L(X,\overline{Z}) &= -\sum_{i\in I} f_i x_i + \sum_{j\in J}\Big(\sum_{i\in I} p_{ij}x_{ij}\Big)\Big(1-\sum_{i\in I}\bar{z}_{ij}\Big) \\ &\le -\sum_{i\in I} f_i x_i + \sum_{j\in J} p_j \max_{i\in I_j} x_i = -\sum_{i\in I} f_i x_i + \sum_{j\in J} p_j \max_{i|x_i=1}(1-h_{ij}) \\ &= \sum_{j\in J} p_j - \Big(\sum_{i\in I} f_i x_i + \sum_{j\in J}\min_{i|x_i=1} p_j h_{ij}\Big). \end{aligned}$$

Hence, we deduce that

$$\max_{(x_i)}\Big\{\sum_{j\in J}p_j - \Big(\sum_{i\in I} f_i x_i + \sum_{j\in J}\min_{i|x_i=1}p_j h_{ij}\Big)\Big\} = \sum_{j\in J}p_j - \min_{(x_i)}\Big\{\sum_{i\in I} f_i x_i + \sum_{j\in J}\min_{i|x_i=1}p_j h_{ij}\Big\}$$

is the required upper bound. The proof of the theorem is complete. □

This form of the upper bound implies that its calculation reduces to solving the well-known facility location problem [7, 13]. Thus, an algorithm for calculating the upper bound includes the two stages: At the first stage, we construct the original data (determine the matrix (h_{ij}), $i\in I$, $j\in J$) of the facility location problem under study, and, at the second stage, we determine its optimal solution $\big(x_i^*\big)$, $i\in I$.

The procedure for constructing (h_{ij}), $i\in I$, $j\in J$, consists of n steps. At step j, for a fixed element $j\in J$, we have m substeps of the same type. At substep i, we calculate h_{ij}. In the beginning, put $h_{ij}=0$. If $i\notin A_j$ or $f_i=\infty$ then put $h_{ij}=1$. Suppose that $i\in A_j$ and $f_i\neq\infty$. First of all, for every $s\in J$, calculate the best element i_s of the set $A_s\setminus N(i)$ with respect to $\succ_s$; i.e., the element satisfying $i_s \not\prec_s k$ for every $k\in A_s\setminus N(i)$. If $A_s\setminus N(i)=\varnothing$ then put $i_s=i$. Furthermore, for every $k\in I$, $g_k\neq\infty$, such that $k\succ_j i$, determine $s\in J$ satisfying $i\not\prec_s i_s$ and $k\succ_s i$, which corresponds to constructing the set $J(k,i)=\{s\in J \mid i\not\prec_s i_s,\, k\succ_s i\}$. Simultaneously we verify the validity of the inequality

$$g_k \le \sum_{s|i\not\prec_s i_s,\,k\succ_s i} p_{ks} - \sum_{s|i_s\succ_s i,\,k\succ_s i}\max\Big\{0;\max_{l|i\succ_j l,\,g_l\neq\infty,\,k\succ_s l\succ_s i}(p_{ls}-p_{ks})\Big\}.$$

If this holds for some $k\in I$ satisfying $g_k\neq\infty$ and $k\succ_j i$ then put $h_{ij}=1$. After that the next substep starts.

It is not hard to see that the fulfilment of one substep requires time $O(mn)$. Thus, the time complexity of the first stage of the algorithm for calculating the upper bound is $O(m^2n^2)$.

At the second stage, in order to solve the facility location problem, we can use a whole series of algorithms [7, 13] based on the ideas of implicit exhaustive search and local search methods, including commercial software for solving integer linear programming problems.

3. AN ALGORITHM FOR CONSTRUCTING AN APPROXIMATE SOLUTION TO (1’), (2)–(9)

We noted above that every admissible solution X to problem L' may serve as an approximate solution to (1′), (2)–(9). The value $L(X,\overline{Z})$ of the objective function on the corresponding admissible solution $(X,\overline{Z})$ to (1′), (2)–(9) is uniquely determined by X. Moreover, the admissible solution $X=((x_i)(x_{ij}))$ itself is uniquely determined by the $(0,1)$-vector (x_i). Thus, we can identify an admissible solution to (1′), (2)–(9) with a $(0,1)$-vector (x_i). Every vector (x_i) of this type uniquely determines the value $L(X,\overline{Z})$ of the objective function of the problem on the corresponding admissible solution $(X,\overline{Z})$.

Observe that simultaneously with calculating an upper bound for the optimal value of the objective function of (1′), (2)–(9) we determine the optimal solution (x_i^*) to the corresponding facility location problem. We can regard this optimal solution as an initial approximate solution to (1′), (2)–(9). However, since we are interested in constructing an approximate solution with a nearly optimal value of the objective function, we consider a procedure for improving the initial approximate solution.

This procedure is based on local search in some particular neighborhood. In the case of local search in the set of $(0,1)$-vectors $x = (x_i)$, it is customary to use the neighborhood

$$N(x) = \{y \mid d(x,y) \le 1 \text{ or } d(x,y) \le 2,\ d(0,x) = d(0,y)\},$$

where $d(x,y)$ is the Hamming distance equal to the number of unequal coordinates of the $(0,1)$-vectors x and y. The elements of this set result from x by changing the values of one or two coordinates whose sum is equal to 1. In our case, searching in $N(x)$ can be complicated; calculating for every $y \in N(x)$ the corresponding admissible solution $(X, \overline{Z})$ to the problem (1′), (2)–(9) can turn out a rather laborious procedure at least because every time it requires us to solve two integer linear programming problems. Thus, given a $(0,1)$-vector x, define a neighborhood $N_0(x) \subset N(x)$ containing only relatively few most promising variations of x. As the main characteristic whose value is an estimate how expedient it is to include (or exclude) products of this type in a prospective solution, we use the profit on the products of this type made by the leader company. For an admissible solution $X = ((x_i)(x_{ij}))$, this quantity for $i \in I$, $x_i = 1$, is equal to

$$\sum_{j \in J} p_{ij} x_{ij} - f_i.$$

The set $N_0(x)$ of promising local variations of x is constructed as follows: The number of elements in $N_0(x)$ is equal to the number of elements in $k \in I$ satisfying $f_k \ne \infty$. Let $I_0(x)$ denote the set $\{i \in I \mid x_i = 1\}$, and, for every $k \in I$, $f_k \ne \infty$, define a $(0,1)$-vector $x^k = (x_i^k)$ as follows:

If $x_k = 1$ then put $x_i^k = x_i$ for $i \ne k$ and $x_k^k = 0$.

If $x_k = 0$ then put $x_i^k = x_i$ for $i \ne k$ and $x_k^k = 1$. Furthermore, construct an admissible solution $X^k = ((x_i^k)(x_{ij}^k))$ and calculate the profit on type k products equal to

$$\sum_{j \in J} p_{kj} x_{kj}^k - f_k.$$

If it is nonnegative then, for every $i \in I_0(x)$, calculate the profit

$$\sum_{j \in J} p_{ij} x_{ij}^k - f_i.$$

If, for every $i \in I_0(x)$, the profit is nonnegative, i.e., the appearance of type k products on the market does not make other types unprofitable, then assume that the vector (x_i^k) is constructed. However, if, for some $i \in I_0(x)$, the profit is negative then find $i_0 \in I_0(x)$ for which this quantity is smallest, put $x_{i_0}^k = 0$, and complete the construction of (x_i^k).

If

$$\sum_{j \in J} p_{kj} x_{kj}^k - f_k < 0$$

then choose among $l \in I_0(x)$ an element l_0 such that the removal of type l_0 products from the market leads to the maximal increase in the profit on type k products. To this end, for every $l \in I_0(x)$, consider the $(0,1)$-vector (y_i^l), where $y_i^l = x_i^k$ for $i \ne l$ and $y_l^l = 0$. Construct the admissible solution $Y^l = ((y_i^l)(y_{ij}^l))$, calculate the profit

$$\sum_{j \in J} p_{kj} y_{kj}^l - f_k$$

on the type k products, and choose l_0 for which this quantity is greatest. After that put $x^k_{l_0} = 0$ and assume that (x^k_i) is constructed.

The algorithm for improving an original approximate solution $x^* = (x^*_i)$ amounts to local ascent search in the neighborhood $N_0(x)$ from $x^* = (x^*_i)$ to a local maximum of $L(X, \overline{Z})$. The algorithm consists of a preliminary step and a number of main steps of the same type.

At the preliminary step, we have the vector (x^*_i). The step amounts to constructing from this vector an admissible solution $(X, \overline{Z})$ and evaluating $L(X, \overline{Z})$. To this end, from (x^*_i) we construct the admissible solution $X = ((x^*_i)(x^*_{ij}))$ to problem L'; then, for the fixed X, determine the optimal solution $\widetilde{Z}$ to problem F; and finally, for the fixed X and $\widetilde{Z}$, determine the optimal solution $\overline{Z}$ to the auxiliary problem (10)–(14) and evaluate $L(X, \overline{Z})$. After that we assume that the initial admissible solution $(X, \overline{Z})$ is constructed, and start the first main step.

At every main step, we have the vector $x = (x_i)$, the corresponding admissible solution $(X, \overline{Z})$, and the value L of the objective function of the problem (1′), (2)–(9) at this solution. The step amounts to successively inspecting the elements of $N_0(x)$ and choosing among them the solution which yields a better admissible solution than the current $(X, \overline{Z})$. In order to find this solution for every $k \in I$, $f_k \neq \infty$, we construct $x^k = (x^k_i)$ and determine the corresponding admissible solution $(X^k, \overline{Z}^k)$ and the value L^k of the objective function. To this end, from (x^k_i) we construct the admissible solution $X^k = ((x^k_i)(x^k_{ij}))$ to problem L'; then, for the fixed X^k, determine the optimal solution $\widetilde{Z}^k$ to problem F; and finally, for fixed X^k and $\widetilde{Z}^k$, we determined the optimal solution $\overline{Z}^k$ to (10)–(14) and evaluate L^k. If $L \geq L^k$ then we consider the next element $k \in I$, $f_k \neq \infty$, and construct a new vector $x^k \in N_0(x)$; otherwise we put $x = x^k$, $X = X^k$, $\overline{Z} = \overline{Z}^k$, $L = L^k$, and start the next step. If at this step $L \geq L^k$ for all $k \in I$, $f_k \neq \infty$, then the algorithm stops, and $(X, \overline{Z})$ is the required approximate solution to the problem.

The running time of the algorithm depends on the number of steps necessary to construct the local optimum and the time complexity of every step is determined by the time it takes to construct the neighborhood $N_0(x)$, which is $O(mn)$, and the time necessary to solve problems F and (10)–(14). Meanwhile, observe that problems F and (10)–(14) are mixed linear programming problems since in both problems the variables z_{ij}, $i \in I$, $j \in J$, may be assumed nonnegative. Their values 0 and 1 guarantee the restrictions (7) and (11).

4. A NUMERICAL EXAMPLE

This example of a decision-making problem for the leader company includes 12 consumers and 12 types of products which can be offered to the market by both companies. The leader company makes the types of products with indices from 1 to 6, while the follower company, from 7 to 12. Each pair of products with indices i and $i + 1$, where i is odd, corresponds to products with the same consumer properties, but sold at different prices. Products with odd indices are sold at low prices, and those with even indices are sold at high prices. The fixed cost f_i of the leader company on the type i products, $1 \leq i \leq 6$, are equal to 40 for odd i and 35 for even i. Similarly, the fixed cost g_i of the follower company on the type i products, $7 \leq i \leq 12$, are equal to 35 for odd i and 30 for even i.

The sets A_j, $j = 1, \dots, 12$, of admissible types of products for each consumer contain the following elements in the order of the corresponding consumer's preferences:

$$A_1 = \{3, 7, 4, 8, 5\}, \qquad A_7 = \{5, 7, 6, 8\},$$
$$A_2 = \{3, 9, 10, 4, 1, 2\}, \qquad A_8 = \{11, 12, 7, 8, 1, 2, 5\},$$
$$A_3 = \{9, 10, 1, 2\}, \qquad A_9 = \{5, 11, 12, 6\},$$
$$A_4 = \{9, 3, 10, 4\}, \qquad A_{10} = \{3, 7, 8, 4, 5\},$$
$$A_5 = \{7, 8, 5, 6\}, \qquad A_{11} = \{1, 11, 2, 5, 12, 6\},$$
$$A_6 = \{7, 5, 8, 6, 1\}, \qquad A_{12} = \{5, 3, 9, 6, 10, 4\}.$$

For the types of products in A_j, $j = 1, \dots, 12$, the corresponding revenue of the manufacturer is determined by

$$p_{3,1} = 10,\ p_{4,1} = 12,\ p_{5,1} = 12; \qquad p_{7,1} = 9.6,\ p_{8,1} = 12;$$
$$p_{1,2} = 15,\ p_{2,2} = 20,\ p_{3,2} = 14,\ p_{4,2} = 16.8; \qquad p_{9,2} = 14.4,\ p_{10,2} = 16;$$
$$p_{1,3} = 18,\ p_{2,3} = 24; \qquad p_{9,3} = 14.4,\ p_{10,3} = 16;$$
$$p_{3,4} = 10,\ p_{4,4} = 12; \qquad p_{9,4} = 10.8,\ p_{10,4} = 12;$$
$$p_{5,5} = 18,\ p_{6,5} = 22.5; \qquad p_{7,5} = 12,\ p_{8,5} = 15,$$
$$p_{1,6} = 18,\ p_{5,6} = 14.4,\ p_{6,6} = 18; \qquad p_{7,6} = 14.4,\ p_{8,6} = 18;$$
$$p_{5,7} = 8.4,\ p_{6,7} = 10.5, \qquad p_{7,7} = 9.6,\ p_{8,7} = 12;$$
$$p_{1,8} = 12,\ p_{2,8} = 16,\ p_{5,8} = 18; \qquad p_{7,8} = 14.4,\ p_{8,8} = 18,$$
$$p_{11,8} = 15,\ p_{12,8} = 18;$$
$$p_{5,9} = 16.8,\ p_{6,9} = 21; \qquad p_{11,9} = 15,\ p_{12,9} = 18;$$
$$p_{3,10} = 10,\ p_{4,10} = 12,\ p_{5,10} = 12; \qquad p_{7,10} = 8,\ p_{8,10} = 10;$$
$$p_{1,11} = 18,\ p_{2,11} = 24,\ p_{5,11} = 24,\ p_{6,11} = 30; \qquad p_{11,11} = 21,\ p_{12,11} = 25.2;$$
$$p_{3,12} = 16,\ p_{4,12} = 19.2,\ p_{5,12} = 14.4,\ p_{6,12} = 18; \qquad p_{9,12} = 16.2,\ p_{10,12} = 18.$$

Using these quantities, we can easily calculate the profit made by each company for every decision (x_i) of the leader company. For instance, if the leader company makes the decision (x_i) with $x_3 = 1$, $x_5 = 1$, and $x_i = 0$ for $i \neq 3$ and $i \neq 5$ then the corresponding optimal solution for the follower company is the vector $(\bar{z}_i)$ with $\bar{z}_7 = 1$ and $\bar{z}_i = 0$ for $i \neq 7$. Meanwhile, the leader company serves the consumers with indices 1, 2, 4, 7, 9, 10, 11, and 12, which yield to it 107.6 in revenue. The follower company serves the consumers with indices 5, 6, 8, and collects 40.8 in revenue. The profit of the leader company is equal to 27.6, and that of the follower company is 5.8. Observe that, in the solutions considered, consumer 3 remains unserved.

The sets I_j and coefficients p_j, $j = 1, \dots, 12$, necessary for calculating an upper bound for the profit of the leader company in this example are

$$I_1 = \{3,4\},\quad p_1 = 12,\qquad I_7 = \{5,6\},\quad p_7 = 10.5,$$
$$I_2 = \{3,4\},\quad p_2 = 16.8,\qquad I_8 = \{1\},\quad p_8 = 12,$$
$$I_3 = \{1,2\},\quad p_3 = 24,\qquad I_9 = \{5,6\},\quad p_9 = 21,$$
$$I_4 = \{3,4\},\quad p_4 = 12,\qquad I_{10} = \{3,4\},\quad p_{10} = 12,$$
$$I_5 = \{5\},\quad p_5 = 18,\qquad I_{11} = \{1,2,5\},\quad p_{11} = 24,$$
$$I_6 = \{5\},\quad p_6 = 14.4,\qquad I_{12} = \{3,5,6\},\quad p_{12} = 18.$$

The optimal value of the objective function of the corresponding facility location problem is equal to 111; thus, the value of the upper bound is $UB = 194.7 - 111 = 83.7$. This optimal value is attained at the solution $\left(x_i^*\right)$ equal to $(0,0,0,1,1,0)$. Note that henceforth for the solutions (x_i) we only indicate the values of the first six components since the rest of them are always zero.

The algorithm for improving the original approximate solution $x = (0,0,0,1,1,0)$ consists of a preliminary step and (in this case) three main steps.

At the preliminary step, we use $x = (0,0,0,1,1,0)$ to determine $\bar{z} = (1,0,0,1,0,0)$ and evaluate the objective function $L = -11.4$. By analogy with (x_i), for solutions $(\bar{z}_i)$, we only indicate the values of the last six components since the first six are always zero.

At the first main step, we have $x = (0,0,0,1,1,0)$ with $L = -11.4$. The step starts with constructing $x^1 = (1,0,0,1,1,0)$, calculating $\bar{z}^1 = (1,0,0,1,0,0)$, and evaluating $L^1 = -57.4$. Since $L^1 < L$, we continue by constructing $x^2 = (0,1,0,1,1,0)$, calculating $\bar{z}^2 = (1,0,0,1,0,0)$, and evaluating $L^2 = -46.4$. Since $L^2 < L$, we consider the next solution $x^3 = (0,0,1,0,1,0)$. For this vector, we calculate $\bar{z}^3 = (1,0,0,0,0,0)$ and evaluate $L^3 = 27.6$. Since $L^3 > L$, the second step starts.

At the second main step, $x = (0, 0, 1, 0, 1, 0)$ with $L = 27.6$. The step amounts to a successive construction of x^k, calculation of the corresponding $\bar{z}^k$, and evaluation of L^k. This yields

$$x^1 = (1, 0, 1, 0, 1, 0), \qquad L^1 = -0.4,$$
$$x^2 = (0, 1, 1, 0, 1, 0), \qquad L^2 = 16.6,$$
$$x^3 = (0, 0, 0, 0, 1, 0), \qquad L^3 = 38.$$

Since $L^3 > L$, the third step starts.

At the third step, $x = (0, 0, 0, 0, 1, 0)$ with $L = 38$. We consider the following vectors x^k and the corresponding values of the objective function L^k:

$$x^1 = (1, 0, 0, 0, 1, 0), \qquad L^1 = -8,$$
$$x^2 = (0, 1, 0, 0, 1, 0), \qquad L^2 = 3,$$
$$x^3 = (0, 0, 1, 0, 1, 0), \qquad L^3 = 27.6,$$
$$x^4 = (0, 0, 0, 1, 1, 0), \qquad L^4 = -11.4,$$
$$x^5 = (0, 0, 0, 0, 0, 0), \qquad L^5 = 0,$$
$$x^6 = (0, 0, 0, 0, 0, 1), \qquad L^6 = -24.5.$$

Since $L^k \leq L$ for every $k = 1, \dots, 6$, the algorithm stops, and $x = (0, 0, 0, 0, 1, 0)$ is the required approximate solution.

The solution $x = (0, 0, 0, 0, 1, 0)$ for the leader company corresponds to the optimal solution $\bar{z} = (0, 1, 0, 1, 0, 0)$ for the follower company. Meanwhile, the leader company serves the consumers with indices 6, 7, 9, 11, and 12, making the profit of 38, and the follower company captures the remaining consumers and makes the profit of 39.

Note that the resulting approximate solution, as it is easy to verify, is not only a local maximum, but also the optimal solution in this example of a decision-making problem for the leader company.

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