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TWO-LEVEL PROGRAMMING PROBLEMS

On the Competitive Facility Location Problem with a Free Choice of Suppliers

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Abstract—We consider a mathematical model from the class of competitive sequential facility location problems. In these problems, the competitors sequentially open their facilities, and each side aims to "capture" the consumers and maximize its profits. In the proposed model, we consider a situation of a "free" choice by each side of an open facility to service a customer. The model is formulated as a bilevel integer programming problem. We show that the problem of finding an optimal noncooperative solution can be represented as a maximization problem for a pseudo-Boolean function. We propose an algorithm for constructing an admissible noncooperative solution for fixed values of the variables in this pseudo-Boolean function. We also propose a method for constructing an upper bound on the maximal value of the pseudo-Boolean function on subsets of solutions defined by partial $(0, 1)$-vectors.

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1. INTRODUCTION

We study a problem similar to the ones considered in [1–3]. In those models, unlike the classical facility location problem [4, 5], there are two competitive sides that sequentially open their facilities. These sides, usually called Leader and Follower [6], aim to capture the consumers in order to achieve their goals. The possibility to capture a given consumer for one of the sides depends on the consumer's preferences that allow him to choose the most preferable facility among the ones that the sides have opened.

The Leader's purpose in this competition is to find a set of facilities that lets him get maximal profits under the condition that a part of the consumers will be captured by the Follower. The Follower's problem is to open a set of facilities that maximizes his profits given that he knows the set of facilities open by the Leader.

Formally speaking, the competitive sequential facility location problem is a bilevel integer programming problem [7–10] that includes an upper level problem (the Leader's problem) and a lower level problem (the Follower's problem). The form of these problems depends on the additional assumptions used in constructing the model. These assumptions deal with the rules used to capture consumers by one of the sides and with the rules used by Leader and Follower in choosing an open facility to service a given consumer.

The work [1] has considered a model where both Leader and Follower use a "hard" choice rule for choosing a supplier to service a consumer. This rule presupposes that the facility most preferable for a given consumer is selected to service this consumer. One can assume that when this rule applies, the consumer himself chooses an open facility to be serviced there. The works [2, 3] consider a model where the Leader uses a hard choice rule, while the Follower has a free choice of facility to service a consumer. In free choice, we suppose that if the Follower captures a consumer, he then himself chooses a supplier for this specific consumer. Such a supplier can be any facility open by the Follower that is more preferable for this consumer than any facility open by the Leader.

In this work, we formulate and study a model where both Leader and Follower have free choice of supplier. In this case the side that has captured a given consumer can use any open facility to service it given that this facility is no worse from this consumer's point of view than any facility open by the other side. The resulting problem presents additional obstacles for study since for this model we cannot avoid dependence of the Leader's constraints on the optimal solution of the Follower's problem.

An important characteristic feature of the model in question, similar to the previously considered versions of the competitive facility location problem, is in the need to refine the notion of an optimal solution. This is due to the fact that an optimal solution for the Follower's problem may not be unique, and this leads to uncertainty in computing the value of the Leader's objective function. For the problem we consider here, similar to the problems from [1–3] we introduce the notion of an optimal noncooperative solution. Using such solutions as optimal lets us correctly pose the problem of finding optimal solutions for the model in question.

The works [1–3] have proposed an approach to constructing algorithms for solving competitive sequential facility location problems that presupposes the following two conditions. First, we have to be able to distinguish a relatively small part of the Leader's problem's variables and represent the competitive location problem as a maximization problem for some pseudo-Boolean function [11] of these variables. For this condition to hold, we have to show that for every $(0, 1)$-vector of the selected variables, the corresponding admissible noncooperative solutions yield the same value of the objective function in the studied competitive location problem, which we take as the value of the pseudo-Boolean function. Besides, we also have to have a way to efficiently compute this value and find at least one admissible noncooperative solution. Second, we have to be able to efficiently compute an upper bound on the values of the resulting pseudo-Boolean function on subsets of the set of $(0, 1)$-vectors. As such subsets, we consider subsets defined by partial $(0, 1)$-vectors. Each such subset consists of a set of $(0, 1)$-vectors with fixed values of certain components. Besides, we suppose that simultaneously with computing the upper bound we also find a $(0, 1)$-vector from the subset in question that yields a "good" lower bound on the maximal value of the pseudo-Boolean function on this subset.

In the present work, we show that in the case of the competitive sequential facility location problem with free choice of suppliers, i.e., the problem we consider in this work, both of the above conditions hold. We construct a pseudo-Boolean function with as many variables as there are possible locations for the Leader's facilities. To compute the value of this function and construct the corresponding admissible noncooperative solution, we have to solve two integer linear programming problems. The first is the Follower's problem, and the second is an auxiliary problem of the same dimension that we construct here. Similar to the results of [1–3], we propose a method for computing upper bounds for the proposed pseudo-Boolean function. For every partial solution, we construct an estimation problem whose objective function's optimal value yields an upper bound, and the optimal solution generates a "good" admissible noncooperative solution.

The paper consists of three main sections. Section 2 shows the formulation of the competitive sequential facility location problem as a bilevel integer programming problem. In Section 3, we introduce the notion of an optimal noncooperative solution for the problem and reduce the problem of finding such a solution to a maximization problem for a pseudo-Boolean function. Section 4 is devoted to a method for computing upper bounds for the values of this pseudo-Boolean function on subsets of $(0, 1)$-vectors defined by partial $(0, 1)$-vectors.

2. PROBLEM SETTING

To formally pose the competitive sequential facility location problem with free choice of suppliers, we first introduce notation and formulate the necessary assumptions.

Similar to the classical facility location problem, we denote by $I = \{1, \dots, m\}$ the set of facilities (possible locations for open facilities); by $J = \{1, \dots, n\}$, the set of consumers.

We assume that a facility $i \in I$ can be opened both by the Leader and the Follower. Therefore, for every $i \in I$ we assume that the values f_i and g_i, i.e., fixed costs to open facility i for the Leader and the Follower respectively, are known. If for some reason either the Leader or the Follower cannot open a facility i, we let $f_i = \infty$ or $g_i = \infty$.

For every $i \in I$ and $j \in J$, we denote by p_{ij} the profit earned by facility i opened by the Leader in servicing consumer j; by q_{ij}, the profit of facility i opened by the Follower in servicing consumer j.

We will assume that either the Leader or the Follower capture a consumer $j \in J$ depending on the preferences of this consumer. We suppose that the preferences of a consumer $j \in J$ are given by an ordering $\succ_j$ on the set I. For $i, k \in I$, relation $i \succ_j k$ means that if both facilities i and k are open, consumer $j \in J$ prefers facility i. The relation $i \succcurlyeq_j k$ means that either $i \succ_j k$ or $i = k$.

Let $I_0 \subset I$. For every $j \in J$ we denote by $i_j(I_0)$ an element $i_0 \in I_0$ such that $i_0 \succcurlyeq_j i$ for every $i \in I_0$. if $I_0 = \{i \in I \mid w_i = 1\}$, where $w = (w_i)$, $i \in I$, is a $(0,1)$-vector, we will also use notation $i_j(w)$ for the element $i_j(I_0)$. If w is the zero vector, we assume that $i_j(w) = 0$, and $i \succ_j 0$ for every $i \in I$.

To find which side captures a consumer $j \in J$ we use the following rule. Let unit components of the $(0,1)$-vector $x = (x_i)$, $i \in I$, denote the facilities opened by the Leader, and let unit components of the $(0,1)$-vector $z = (z_i)$, $i \in I$, denote the facilities opened by the Follower. We assume that a consumer $j \in J$ will be captured by the Leader if $i_j(x) \succcurlyeq_j i_j(z)$ and by the Follower if $i_j(z) \succ_j i_j(x)$.

When choosing a facility to service a captured consumer $j \in J$, we assume that both Leader and Follower use the *free choice* rule. This means that if a consumer j is captured by the Leader, he can assign any facility $i \in I$ open by the Leader to service this consumer given that $i \succcurlyeq_j i_j(z)$. Similarly, if a consumer j is captured by the Follower, he can assign any facility $i \in I$ open by the Follower to service this consumer given that $i \succ_j i_j(x)$.

The purpose of both Leader and Follower is to maximize their corresponding profit, which is the sum of all profits received by facilities opened by them. We assume that the profit of each open facility equals the sum of profits received from consumers serviced by this facility less the fixed costs spent to open this facility.

We introduce the following variables similar to the variables of the classical facility location problem:

x_i is a variable equal to one if the Leader opens facility $i \in I$ and zero otherwise;

x_{ij} is a variable equal to one if facility $i \in I$ opened by the Leader is assigned to service consumer $j \in J$ and zero otherwise;

z_i is a variable equal to one if the Follower opens facility $i \in I$ and zero otherwise;

z_{ij} is a variable equal to one if facility $i \in I$ opened by the Leader is assigned to service consumer $j \in J$ and zero otherwise.

With these variables, the sequential competitive facility location problem with free choice of suppliers can be formulated as the following bilevel integer programming problem:

$$\max_{(x_i),(x_{ij})} \left\{ -\sum_{i \in I} f_i x_i + \sum_{j \in J} \sum_{i \in I} p_{ij} x_{ij} \right\}, \tag{1}$$

$$\widetilde{z}_i + \sum_{k | i \succcurlyeq_j k} x_{kj} \leqslant 1, \quad i \in I, \quad j \in J; \tag{2}$$

$$x_i \geqslant x_{ij}, \quad i \in I, \quad j \in J; \tag{3}$$

$$x_i, x_{ij} \in \{0,1\}, \quad i \in I, \quad j \in J; \tag{4}$$

$(\widetilde{z}_i), (\widetilde{z}_{ij})$ is the optimal solution of the following problem: (5)

$$\max_{(z_i),(z_{ij})} \left\{ -\sum_{i\in I} g_i z_i + \sum_{j\in J}\sum_{i\in I} q_{ij} z_{ij} \right\}, \tag{6}$$

$$x_i + \sum_{k|i \succcurlyeq_j k} z_{kj} \leqslant 1, \quad i \in I, \quad j \in J; \tag{7}$$

$$z_i \geqslant z_{ij}, \quad i \in I, \quad j \in J; \tag{8}$$

$$z_i, z_{ij} \in \{0,1\}, \quad i \in I, \quad j \in J. \tag{9}$$

We denote the upper level problem (1)–(4) by $\mathcal{L}$ and the lower level problem (5)–(9), by $\mathcal{F}$. For problem (1)–(9) in general we will use notation $(\mathcal{L},\mathcal{F})$, and we will assume that the objective function (1) of problem $\mathcal{L}$ also serves as the objective function of problem $(\mathcal{L},\mathcal{F})$.

As for the optimal solution $((\widetilde{z}_i), (\widetilde{z}_{ij}))$ of problem $\mathcal{F}$, we will assume that for every $i \in I$, if $\sum_{j\in J} \widetilde{z}_{ij} = 0$ then $\widetilde{z}_i = 0$. Note that for $i \in I$ this condition holds automatically if $g_i > 0$. We will also assume that if (x_i) is a zero vector than the optimal solution $((\widetilde{z}_i), (\widetilde{z}_{ij}))$ of problem $\mathcal{F}$ is nonzero.

The objective function (1) of problem $\mathcal{L}$ shows the value of the profit obtained by the Leader. Inequalities (2) prohibit to use for servicing a consumer those facilities opened by the Leader that are less preferable for this consumer than facilities opened by the Follower. The same inequalities guarantee that in order to service each consumer one can select only one facility opened by the Leader. Constraint (3) shows that in order to service consumers we can only use open facilities. The objective function and constraints of problem $\mathcal{F}$ have a similar meaning. In particular, constraints (7) show that if $x_i = 1$ for some $i \in I$ then $z_{ij} = 0$ for every $j \in J$ and, consequently, due to the remark above we have $\widetilde{z}_i = 0$.

3. OPTIMAL SOLUTIONS FOR THE PROBLEM

We call a pair $(X, \widetilde{Z})$, where $X = ((x_i),(x_{ij}))$ is an admissible solution of problem $\mathcal{L}$ for a given vector $(\widetilde{z}_i)$, and $\widetilde{Z} = ((\widetilde{z}_i),(\widetilde{z}_{ij}))$ is an optimal solution of problem $\mathcal{F}$ for a given vector (x_i), an *admissible solution* of problem $(\mathcal{L},\mathcal{F})$.

We denote by $L(X,\widetilde{Z})$ the value of the objective function for problem $(\mathcal{L},\mathcal{F})$ on an admissible solution $(X,\widetilde{Z})$, and by $F(Z)$ the value of the objective function for problem $\mathcal{F}$ on the admissible solution Z.

For a fixed vector $x = (x_i)$, we distinguish among admissible solutions $(X,\widetilde{Z})$ of problem $(\mathcal{L},\mathcal{F})$ the solutions that are "good" from the Leader's point of view. We call an admissible solution $(\widetilde{X},\widetilde{Z})$, where $\widetilde{X} = ((x_i),(\widetilde{x}_{ij}))$, a *strong admissible solution* of problem $(\mathcal{L},\mathcal{F})$ if $L(\widetilde{X},\widetilde{Z}) \geqslant L(X,\widetilde{Z})$ for every admissible solution $(X,\widetilde{Z})$, where $X = ((x_i),(x_{ij}))$. It is clear that an admissible solution $(\widetilde{X},\widetilde{Z})$, $\widetilde{X} = ((x_i),(\widetilde{x}_{ij}))$, $\widetilde{Z} = ((\widetilde{z}_i),(\widetilde{z}_{ij}))$, will be strong if for every $j \in J$ equality $x_{ij} = 1$ holds for such $i \in I$ that

$$p_{ij} = \max_{k|k \succ_j i_j(\widetilde{z})} p_{kj} x_k,$$

where $\widetilde{z} = (\widetilde{z}_i)$.

A strong admissible solution $(\overline{X},\overline{Z})$, where $\overline{X} = ((x_i),(\overline{x}_{ij}))$, is called an *admissible noncooperative solution* of problem $(\mathcal{L},\mathcal{F})$ if $L(\overline{X},\overline{Z}) \leqslant L(\widetilde{X},\widetilde{Z})$ for every strong admissible solution $(\widetilde{X},\widetilde{Z})$, where $\widetilde{X} = ((x_i),(\widetilde{x}_{ij}))$. For a fixed vector $x = (x_i)$, since the optimal solution of problem $\mathcal{F}$ is not unique, an admissible noncooperative solution $(\overline{X},\overline{Z})$, where $\overline{X} = ((x_i),(\overline{x}_{ij}))$, is also not uniquely defined. However, for all admissible noncooperative solutions corresponding to vector

$x = (x_i)$ the value of the objective function in problem $(\mathcal{L}, \mathcal{F})$ will be the same. Consequently, the problem of finding an admissible noncooperative solution with maximal value of the objective function is correctly posed. We call an admissible noncooperative solution (X^*, Z^*) of problem $(\mathcal{L}, \mathcal{F})$ an *optimal noncooperative solution* if $L(X^*, Z^*) \geqslant L(\overline{X}, \overline{Z})$ for every admissible noncooperative solution $(\overline{X}, \overline{Z})$.

Since any $(0,1)$-vector x uniquely defines the value of the objective function for problem $(\mathcal{L}, \mathcal{F})$ on the corresponding admissible noncooperative solution, we can represent the problem of finding an optimal noncooperative solution as a maximization problem for some pseudo-Boolean function $f(x)$. The value of this function on the vector x is the value of the objective function for problem $(\mathcal{L}, \mathcal{F})$ on an admissible noncooperative solution corresponding to vector x.

Let us consider the problem of efficiently constructing, for a given $(0,1)$-vector $x = (x_i)$ the corresponding admissible noncooperative solution $(\overline{X}, \overline{Z})$ for problem $(\mathcal{L}, \mathcal{F})$.

We formulate an auxiliary problem that lets us construct, for a given $(0,1)$-vector $x = (x_i)$, the optimal solution $\overline{Z}$ for problem $\mathcal{F}$ such that a strong admissible solution $(\overline{X}, \overline{Z})$, where $\overline{X} = ((x_i), (\overline{x}_{ij}))$, will be an admissible noncooperative solution.

In addition to variables of problem $\mathcal{F}$, we introduce variables u_j, $j \in J$. The nonnegative value u_j equals to the profit the Leader receives from servicing consumer $j \in J$.

Let F^* be the optimal value of the objective function in problem $\mathcal{F}$. We consider the following problem:

$$\min_{(z_i),(z_{ij}),(u_j)} \sum_{j \in J} u_j, \tag{10}$$

$$u_j \geqslant p_{ij} \left(x_i - \sum_{k | k \succ_j i} z_k \right), \quad i \in I, \quad j \in J; \tag{11}$$

$$x_i + \sum_{k | i \succcurlyeq_j k} z_{kj} \leqslant 1, \quad i \in I, \quad j \in J; \tag{12}$$

$$z_i \geqslant z_{ij}, \quad i \in I, \quad j \in J; \tag{13}$$

$$-\sum_{i \in I} g_i z_i + \sum_{i \in I} \sum_{j \in J} q_{ij} z_{ij} \geqslant F^*; \tag{14}$$

$$z_i, z_{ij} \in \{0,1\}, \quad i \in I, \quad j \in J; \tag{15}$$

$$u_j \geqslant 0, \quad j \in J. \tag{16}$$

Theorem 1. *If* $(\overline{Z}, \overline{U})$, *where* $\overline{Z} = ((\overline{z}_i), (\overline{z}_{ij}))$, $\overline{U} = (\overline{u}_j)$, *is an optimal solution of problem* (10)–(16), *then a strong admissible solution* $(\overline{X}, \overline{Z})$, *where* $\overline{X} = ((x_i), (\overline{x}_{ij}))$, *of problem* $(\mathcal{L}, \mathcal{F})$ *is an admissible noncooperative solution of problem* $(\mathcal{L}, \mathcal{F})$.

Proof. Let us first of all note that a strong admissible solution $(\overline{X}, \overline{Z})$, where $\overline{X} = ((x_i), (\overline{x}_{ij}))$, will be an admissible noncooperative solution for problem $(\mathcal{L}, \mathcal{F})$ if $\overline{Z}$ is an optimal solution for the minimization problem with respect to variables (z_i) and (z_{ij}) for the objective function

$$\sum_{j \in J} \max_{i | i \succcurlyeq_j i_j(z)} p_{ij} x_i \tag{17}$$

under constraints (12)–(15). As above, here we have denoted by z the $(0,1)$-vector (z_i).

Indeed, let $Z = ((z_i), (z_{ij}))$ be an admissible solution of problem (17), (12)–(15). Due to constraint (14), this solution will be an optimal solution of problem $\mathcal{F}$. By the solution Z, we construct an admissible solution (X, Z), where $X = ((x_i), (x_{ij}))$, for problem $(\mathcal{L}, \mathcal{F})$ as follows.

For every $j \in J$, if $i_j(z) \succ_j i_j(x)$ we let $x_{ij} = 0$, $i \in I$. If, on the other hand, $i_j(x) \succcurlyeq_j i_j(z)$, we choose an element $i_0 \in I$ such that $x_{i_0} = 1$, $p_{i_0 j} = \max_{i | i \succcurlyeq_j i_j(z)} p_{ij} x_i$, and let

$$x_{ij} = \begin{cases} 1, & \text{if } i = i_0 \\ 0 & \text{otherwise.} \end{cases}$$

By construction, solution (X, Z) will be a strong admissible solution of problem $(\mathcal{L}, \mathcal{F})$. Besides, the value of the objective function (17) on solution Z and the value of the objective function in problem $(\mathcal{L}, \mathcal{F})$ without the fixed value $\left(-\sum_{i \in I} f_i x_i\right)$ coincide on solution (X, Z). Consequently, the strong admissible solution $(\overline{X}, \overline{Z})$ constructed with the optimal solution $\overline{Z}$ of problem (17), (12)–(15) will be an admissible noncooperative solution of problem $(\mathcal{L}, \mathcal{F})$.

Let us now note that problem (17), (12)–(15) can be represented as a minimization problem with respect to variables $(z_i), (z_{ij})$ and (u_j) for the objective function

$$\sum_{j \in J} u_j$$

given that

$$u_j \geqslant p_{ij} x_{ij}, \quad i \succcurlyeq i_j(z), \quad j \in J, \tag{18}$$

and under constraints (12)–(16).

We also note that for every $j \in J$ constraints (18) hold if and only if the following inequalities hold:

$$u_j \geqslant p_{ij} \left(x_i - \sum_{k | k \succ_j i} z_k \right), \quad i \in I.$$

This remark concludes the proof of Theorem 1.

The above proof implies the following algorithm for constructing, by and $(0,1)$-vector $x = (x_i)$, an admissible noncooperative solution $(\overline{X}, \overline{Z})$, where $\overline{X} = ((x_i), (\overline{x}_{ij}))$.

The algorithm consists of two stages.

On stage 1, for a given $(0,1)$-vector x we solve problem $\mathcal{F}$ and find the optimal value of its objective function F^*.

On stage 2, for a given $(0,1)$-vector x we solve problem (10)–(16), find its optimal solution $(\overline{Z}, \overline{U})$, and construct the corresponding strong admissible solution $(\overline{X}, \overline{Z})$, which is an admissible noncooperative solution.

4. UPPER BOUND

Let us study the problem of how to efficiently compute an upper bound on the values of the considered pseudo-Boolean function $f(x)$, $x = (x_i)$, $i \in I$, on subsets of the set of $(0,1)$-vectors. It is convenient to define such subsets with partial $(0,1)$-vectors. A vector $y = (y_i)$, $i \in I$, whose elements take values 0, 1, or an undefined value $*$, is called a *partial* $(0,1)$*-vector* or a *partial solution*. A partial solution divides the variables of function $f(x)$ into variables with defined value 0 or 1 and free variables. For a partial $(0,1)$-vector $y = (y_i)$, we define sets $I^0 = \{i \in I \mid y_i = 0\}$ and $I^1 = \{i \in I \mid y_i = 1\}$. A partial solution $y = (y_i)$ defines the set of $(0,1)$-vectors $x = (x_i)$ such that $x_i = 0$, $i \in I^0$, and $x_i = 1$, $i \in I^1$. We denote this set by $P(y)$.

The value of function $f(x)$ on a $(0,1)$-vector $x \in P(y)$ is the value of the objective function of problem $(\mathcal{L}, \mathcal{F})$ on the noncooperative solution $(\overline{X}, \overline{Z})$ corresponding to vector x. The maximal value of function $f(x)$ on the set $P(y)$ is the optimal value of the objective function in problem $(\mathcal{L}, \mathcal{F})$ with an additional constraint $x_i = y_i$ for $i \in I^0(y) \cup I^1(y)$, i.e., an objective function of the following problem:

$$\max_{(x_i),(x_{ij})} \left\{ -\sum_{i\in I} f_i x_i + \sum_{j\in J}\sum_{i\in I} p_{ij} x_{ij} \right\}, \tag{19}$$

$$\widetilde{z}_i + \sum_{k|i \succcurlyeq_j k} x_{kj} \leqslant 1, \quad i \in I, \quad j \in J; \tag{20}$$

$$x_i \geqslant x_{ij}, \quad i \in I, \quad j \in J; \tag{21}$$

$$x_i = y_i, \quad i \in I^0(y) \cup I^1(y); \tag{22}$$

$$x_i, x_{ij} \in \{0,1\}, \quad i \in I, \quad j \in J; \tag{23}$$

$$(\widetilde{z}_i), (\widetilde{z}_{ij}) \text{ is an optimal solution of problem (6)–(9).} \tag{24}$$

We denote problem (19)–(24) by $\mathcal{L}(y)$ and denote problem (19)–(24), (6)–(9) as a whole by $(\mathcal{L}(y), \mathcal{F})$.

To construct an upper bound on the optimal value of the objective function in problem $(\mathcal{L}(y), \mathcal{F})$, consider the values $p'_{ij} = \max_{k|i \succcurlyeq k} p_{kj}$, $i \in I$, $j \in J$. We denote problem $\mathcal{L}(y)$ where instead of p_{ij} we use the values p'_{ij} by $\mathcal{L}'(y)$, and the objective function of this problem by $L'(X, \widetilde{Z})$.

Note that for a fixed $(0,1)$-vector $x = (x_i)$, $x \in P(y)$, for every strong admissible solution $(\widetilde{X}, \widetilde{Z})$, where $\widetilde{X} = ((x_i), (\widetilde{x}_{ij}))$, for problem $(\mathcal{L}(y), \mathcal{F})$ there exists a strong admissible solution $(\widetilde{X}', \widetilde{Z})$, where $\widetilde{X}' = ((x_i), (\widetilde{x}'_{ij}))$, for problem $(\mathcal{L}'(y), \mathcal{F})$ such that $L(\widetilde{X}, \widetilde{Z}) \leqslant L'(\widetilde{X}', \widetilde{Z})$. Therefore, for every admissible noncooperative solution $(\overline{X}, \overline{Z})$, where $\overline{X} = ((x_i), (\overline{x}_{ij}))$, for problem $(\mathcal{L}(y), \mathcal{F})$ there exists an admissible noncooperative solution $(\overline{X}', \overline{Z}')$, where $\overline{X}' = ((x_i), (\overline{x}'_{ij}))$, of problem $(\mathcal{L}'(y), \mathcal{F})$ such that $L(\overline{X}, \overline{Z}) \leqslant L'(\overline{X}', \overline{Z}')$. Consequently, the optimal value of the objective function in problem $(\mathcal{L}'(y), \mathcal{F})$ yields an upper bound on the set of values of pseudo-Boolean function $f(x)$ on the set $P(y)$.

We note further that for every $j \in J$ values p'_{ij}, $i \in I$, are monotone with respect to the ordering $\succ_j$, i.e., for every $i, k \in I$, $i \succ_j k$, we have $p'_{ij} \geqslant p'_{kj}$. Therefore, in problem $(\mathcal{L}'(y), \mathcal{F})$ we can restrict ourselves to considering only such strong admissible solutions $(\widetilde{X}, \widetilde{Z})$, where $\widetilde{X} = ((x_i), (\widetilde{x}_{ij}))$, that

$$\widetilde{x}_{ij} = \begin{cases} 1, & \text{if } i = i_j(x) \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we can assume that in problem $(\mathcal{L}'(y), \mathcal{F})$ the Leader uses the hard choice rule to select an open facility to service a consumer.

Therefore, we can rewrite problem $(\mathcal{L}'(y), \mathcal{F})$ as follows:

$$\max_{(x_i),(x_{ij})} \left\{ -\sum_{i\in I} f_i x_i + \sum_{j\in J} \left(\sum_{i\in I} p'_{ij} x_{ij} \right) \left(1 - \sum_{i\in I} \widetilde{z}_{ij} \right) \right\},$$

$$\sum_{i\in I} x_{ij} \leqslant 1, \quad j \in J;$$

$$x_i \geqslant x_{ij}, \quad i \in I, \quad j \in J;$$

$$x_i = y_i, \quad i \in I^0(y) \cup I^1(y);$$

$$x_i, x_{ij} \in \{0,1\}, \quad i \in I, \quad j \in J;$$

$$(z_i), (z_{ij}) \text{ is an optimal solution of problem (6)–(9).}$$

For this problem, similar to problems from [1–3] we can formulate an *estimation* problem with the method for constructing a system of subsets I_i, $j \in J$.

The estimation problem is written as follows:

$$\max_{(x_i),(x_{ij})} \left\{ -\sum_{i\in I} f_i x_i + \sum_{j\in J} \sum_{i\in I_j(y)} p'_{ij} x_{ij} \right\},$$

$$\sum_{i\in I} x_{ij} \leqslant 1, \quad j \in J;$$

$$x_i \geqslant x_{ij}, \quad i \in I, \quad j \in J;$$

$$x_i = y_i, \quad i \in I^0(y) \cup I^1(y);$$

$$x_i, x_{ij} \in \{0,1\}, \quad i \in I, \quad j \in J.$$

We denote by $B(X)$ the value of the objective function for this problem on solution $X = ((x_i),(x_{ij}))$; by $X^0 = ((x_i^0),(x_{ij}^0))$, the optimal solution of this problem.

The following theorem can be proven in a way similar to the proof of similar statements from [1–3].

Theorem 2. *If $y = (y_i)$ is a partial solution then for every admissible noncooperative solution $(X, \overline{Z})$ of problem $(\mathcal{L}'(y), \mathcal{F})$ it holds that $L'(X, \overline{Z}) \leqslant B(X^0)$.*

This implies that for every admissible noncooperative solution $(\overline{X}, \overline{Z})$ of problem $(\mathcal{L}(y), \mathcal{F})$ it holds that $L(\overline{X}, \overline{Z}) \leqslant B(X^0)$. Here the $(0,1)$-vector $x^0 = (x_i^0)$ can be viewed as an initial approximate solution for the maximization problem of the pseudo-Boolean function $f(x)$ on the set $P(y)$.

Our representation of the competitive facility location problem with free choice of servicing facility as a maximization problem for a pseudo-Boolean function of the same variables as in other cases of the competitive sequential location problem together with the possibility to compute upper bounds on the values of this function on subsets of $(0,1)$-vectors let us use to solve this problem algorithms constructed for other versions of the competitive sequential facility location problem, in particular, algorithms based on local search [1, 2, 8, 12] and the branch-and-bound method [3].

5. CONCLUSION

In this work, we have studied a new setting for the competitive sequential facility location problem. In the proposed mode, we assume that both the Leader and the Follower use the free choice rule to choose a facility to service consumers. This version of the problem appears to be harder to study since in this case one cannot avoid the presence of elements of the Follower's problem's optimal solution in the constraints of the Leader's problem. However, this case of the competitive sequential facility location problem is very important for further studies. Using the free choice rule for the suppliers lets one formulate meaningful competitive facility location problem settings with bounded volumes of production.

The results obtained in this work, namely results related to representing the problem as a maximization problem for a pseudo-Boolean function of the same variables as for previously considered problems lets us employ for its solution the already existing and tested variety of algorithms that solve competitive sequential facility location problems. A modification of these algorithms in application to the proposed problem deals only with the procedure that computes a "new" pseudo-Boolean function.

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