ISSN 0965-5425, Computational Mathematics and Mathematical Physics, 2009, Vol. 49, No. 6, pp. 994–1009. © Pleiades Publishing, Ltd., 2009.
Original Russian Text © A.V. Kononov, Yu.A. Kochetov, A.V. Plyasunov, 2009, published in Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki, 2009, Vol. 49, No. 6, pp. 1037–1054.

Competitive Facility Location Models

A. V. Kononov, Yu. A. Kochetov, and A. V. Plyasunov
Novosibirsk State University, ul. Pirogova 2, Novosibirsk, 630090 Russia
e-mail: jkochet@math.nsc.ru

Received February 29, 2008; in final form, August 11, 2008

Abstract—Two classes of competitive facility location models are considered, in which several persons (players) sequentially or simultaneously open facilities for serving clients. The first class consists of discrete two-level programming models. The second class consists of game models with several independent players pursuing selfish goals. For the first class, its relationship with pseudo-Boolean functions is established and a novel method for constructing a family of upper and lower bounds on the optimum is proposed. For the second class, the tight PLS-completeness of the problem of finding Nash equilibriums is proved.

DOI: 10.1134/S0965542509060086

Keywords: location problems, local optima, Nash equilibriums, PLS complete problems.

INTRODUCTION

Location problems form a wide class of mathematical operations research models, which is interesting both practically and from the point of view of the combinatorial optimization theory. This problem dates back to Fermat (1601–1665) and Torichelli (1608–1647) (see [1]). As a branch of operations research in its own right, it formed in the 1970–1980s. Presently, there are several monographs on this topic (see [2–6]). Each year, the European workgroup EWGLA (http://www.vub.ac.be/EWGLA/) and the American workgroup SOLA (http://www.ent.ohiou.edu/~thale/sola/sola.html) hold conferences devoted to this topic.

In Soviet Union, the pioneers in the development of this branch of science were V. Cherenin, V. Khachaturov, V. Trubin, and S. Lebedev; in the Siberian Branch of the Academy of Sciences, V. Beresnev, E. Gimadi, and V. Dement'ev were the first to work on such problems. The interest in this kind of problems is mainly due to various applications that arise not only in choosing the facility locations; similar models appear, for example, in problems concerning unification and standardization when the composition of hardware for performing a certain set of tasks is chosen (see [7]). In this case, the objective function is the total cost of the creation and operation of the hardware system or the total performance of the system.

In this paper, we consider two new classes of competitive facility location problems in which several persons (players) simultaneously or one after another make decisions concerning opening facilities for serving clients. If the players make decisions one after another, we have a Stackelberg game, and the corresponding mathematical model can be represented as a discrete two-level programming model. If the decisions are made simultaneously, we have a several person game that want to maximize their own profit. It is assumed that they do not form coalitions, act independently of each other, and pursuit only selfish goals. For the first model in the case of two players, a method for constructing a family of upper and lower bounds on the optimal value of the objective function is proposed, and the relationship with pseudo-Boolean functions is established. For the second class of game models, the concept of the Nash equilibrium is introduced, and its relationship with local optima is established.

The paper is organized as follows. In Section 1, we consider the classical location models in which there is a single decision maker (DM). The relationship of these problems with pseudo-Boolean functions is discussed. It is shown how such functions can be used to transform the initial data so as to minimize the dimension of the problem. In Section 2, we consider the location problem with clients' preferences. In distinction from the preceding problem, here we have two levels at which decisions are made. First, a company opens its facilities. Then, the clients choose their suppliers. It is known that this problem can be reduced to an integer linear program (ILP) and can be equivalently formulated in terms of pseudo-Boolean functions. As in Section 1, it is shown how the dimension of the problem can be decreased. In Section 3, a two-level facility location model is formulated in which two DMs sequentially make decisions concerning opening their facilities. A mathematical statement of this problem in terms of pseudo-Boolean functions is given, and a method for constructing a family of upper and lower bounds is proposed. In Section 4, we consider a game model for several DMs that simultaneously open their facilities for serving clients. The concept of equilibrium solutions is defined, and it is proved that the problem of finding such solutions is tightly PLS complete.

1. SIMPLE LOCATION PROBLEM

In the majority of facility location models, it is assumed that there is a single DM. One may suppose that this is the manager of the company that wants to find an optimal location for its facilities. For the given set of clients $J = \{1, 2, \ldots, n\}$, the DM knows the cost $c_{ij} \geq 0$ of producing and delivering the goods from the jth location to the ith client if a facility is open at site i. The list of possible sites $I = \{1, 2, \ldots, m\}$ is assumed to be finite. For each $i \in I$, the cost $f_i \geq 0$ of opening the facility at this site is given. The problem is to find the set $S \subseteq I$ that makes it possible to serve all the clients at the minimal total cost; that is, we want to find

$$\min_{S \subseteq I}\left\{\sum_{i \in S} f_i + \sum_{j \in J} \min_{i \in S} c_{ij}\right\}.$$

The first term in the objective function is the cost of opening the facilities. The second term represents the production and delivery costs. This problem is known in the literature as the simple plant location problem. It is NP-hard in the strong sense even in the case when the matrix (c_{ij}) satisfies the triangle inequality (see [8]).

There is a close relationship between this problem and the minimization of pseudo-Boolean functions. This fact was first mentioned in the studies by Hammer and Rudeanu. Later, in the works by Beresnev, a novel method for the reduction of the simple location problem to the problem of minimizing pseudo-Boolean functions with positive coefficients at nonlinear terms was proposed. Moreover, the equivalence of these problems was established.

Suppose that, for a certain vector g_i $(i \in I)$, a permutation $i_1, \ldots, i_m$ is known such that $g_{i_1} \leq \ldots \leq g_{i_m}$. Define

$$\begin{aligned} \Delta g_0 &= g_{i_1}, \\ \Delta g_l &= g_{i_{l+1}} - g_{i_l}, \quad 1 \leq l < m, \\ \Delta g_m &= g_{i_m}. \end{aligned}$$

Lemma 1 (see [7]). *For any vector* $z_i \in \{0, 1\}$ $(i \in I, z \neq \{1, \ldots, 1\})$, *it holds that*

$$\min_{i | z_i = 0} g_i = \Delta g_0 + \sum_{l=1}^{m-1} \Delta g_l z_{i_1} \ldots z_{i_l},$$

$$\max_{i | z_i = 0} g_i = \Delta g_m - \sum_{l=1}^{m-1} \Delta g_{m-l} z_{i_{m-l+1}} \ldots z_{i_m}.$$

For the jth column of the matrix (c_{ij}), consider a permutation $i_1^j, \ldots, i_m^j$ such that $c_{i_1^j j} \leq \ldots \leq c_{i_m^j j}$. Using Lemma 1, we represent the objective function of the location problem as the pseudo-Boolean function

$$b(z) = \sum_{i \in I} f_i (1 - z_i) + \sum_{j \in J}\left(\Delta c_{0j} + \sum_{l=1}^{m-1} \Delta c_{lj} z_{i_1^j} \ldots z_{i_l^j}\right).$$

Theorem 1 (see [7]). *The minimization problem for the pseudo-Boolean function* $b(z)$, $z \neq (1, \ldots, 1)$ *and the simple facility location problem are equivalent. For the optimal solutions* z^* and S^* *of these problems, it holds that* $z_i^* = 0 \Leftrightarrow i \in S^*$ $(i \in I)$, *and the values of the objective functions on these solutions are identical.*

Consider the following example:

$$I = J = \{1, 2, 3\}, \quad f_i = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}, \quad c_{ij} = \begin{pmatrix} 0 & 3 & 10 \\ 5 & 0 & 0 \\ 10 & 20 & 7 \end{pmatrix}.$$

The corresponding pseudo-Boolean function is $b(z) = 10(1 - z_1) + 10(1 - z_2) + 10(1 - z_3) + (5z_1 + 5z_1z_2) + (3z_2 + 17z_1z_2) + (7z_2 + 3z_2z_3) = 15 + 5(1 - z_1) + 0(1 - z_2) + 10(1 - z_3) + 22z_1z_2 + 3z_2z_3$. Given this function, we can reconstruct the location problem:

$$I' = I, \quad J' = \{1, 2\}, \quad f'_i = \begin{pmatrix} 5 \\ 0 \\ 10 \end{pmatrix}, \quad c'_{ij} = \begin{pmatrix} 0 & 3 \\ 0 & 0 \\ 22 & 0 \end{pmatrix}.$$

The new problem involves fewer clients: $|J'| < |J|$. Moreover, $f'_2 = 0$. Therefore, the second facility may be assumed to be open in the optimal solution. In other words, the dimension of the new problem is less than that of the original one, and these problems are equivalent.

Therefore, given a simple location problem, we can construct a pseudo-Boolean function with positive coefficients at the nonlinear terms. Different location problems can correspond to the same function. Therefore, before solving the problem, one should first find an equivalent problem of a lower dimension. Such a problem can be found in polynomial time.

Theorem 2. *For a given pseudo-Boolean function with positive coefficients at the nonlinear terms, an equivalent simple location problem with the minimal number of clients can be found in a time that polynomially depends on n and m.*

Proof. Consider an arbitrary pseudo-Boolean function $b(z)$ with positive coefficients at the nonlinear terms

$$b(z) = \sum_{i \in I} \alpha_i z_i + \sum_{l \in L} \beta_l \prod_{i \in I_l} z_i, \quad \beta_l > 0, \quad I_l \subset I, \quad l \in L.$$

The family of subsets $\{I_l\}_{l \in L}$ of the set I with the order relation $I_{l'} < I_{l''} \Leftrightarrow I_{l'} \subset I_{l''}$ forms a partially ordered set. Any sequence of subsets $I_{l_1} < \ldots < I_{l_K}$ is called a *chain*. An arbitrary partitioning of the family $\{I_l\}_{l \in L}$ into nonoverlapping chains induces a matrix of transportation costs (c_{ij}) for a simple location problem. Every element of such a partitioning corresponds to a client. The requirement to find a matrix with the minimum number of clients is equivalent to finding a partitioning of a partially ordered set into the minimum number of nonoverlapping chains. This is a well-known problem the complexity of which polynomially depends on n and m; its solution is based on the constructive proof of Dilworth's theorem (see [9]). This completes the proof of Theorem 2.

The minimization of $b(z)$ is equivalent to the simple location problem, but it also has some new properties. Consider the minimization of this function for continuous variables $z_i \in [0, 1]$ $(i \in I)$. For the simple location problem, such a transition is due to the integrality gap, which can occur arbitrarily close to unity (see [10]). For the function $b(z)$, the integrality gap vanishes! It is easy to verify that there is an integer optimal solution among the optimal solutions of the minimization problem for the function $b(z)$ with continuous variables.

2. LOCATION PROBLEM WITH CLIENTS' PREFERENCES

So far we assumed that there is only one DM who wants to minimize the total production and delivery costs. However, clients often have the possibility to choose suppliers based on their own preferences (see [11]). The minimization of the company's production and delivery costs is not the goal of the clients.

Let the matrix (g_{ij}) specify the clients' preferences on the set I. If $g_{i_1 j} < g_{i_2 j}$, then the client j prefers the facility i_1. To simplify the model, we assume that all the entries in each column of the matrix (g_{ij}) are different. Otherwise, we would have to consider optimistic and pessimistic strategies and introduce additional definitions of the optimal solution of the problem. Thus, the goal of the DM is to choose a subset $S \subseteq I$ that makes it possible to serve all the clients at minimum cost; however, the DM must now take into account the clients' preferences. Let us define the variables of the problem.

$$x_i = \begin{cases} 1, & \text{if a facility opens at location } i, \\ 0, & \text{otherwise}, \end{cases}$$

$$x_{ij} = \begin{cases} 1, & \text{if client } j \text{ is served from location } i, \\ 0, & \text{otherwise}. \end{cases}$$

The mathematical model can be represented by a discrete two-level programming problem (see [11]) that is stated as follows. Find

$$\min_{x_i} \left\{ \sum_{i \in I} f_i x_i + \sum_{j \in J} \sum_{i \in I} c_{ij} x_{ij}^*(x_i) \right\}$$

subject to $x_i \in \{0, 1\}$, $i \in I$, where $x_{ij}^*(x_i)$ is the optimal solution of the auxiliary problem (below, it is called the client problem)

$$\min_{x_{ij}} \sum_{j \in J} \sum_{i \in I} g_{ij} x_{ij}$$

subject to

$$\sum_{i \in I} x_{ij} = 1, \quad x_{ij} \le x_i, \quad x_{ij} \in \{0, 1\}, \quad i \in I, \quad j \in J.$$

As before, the DM's objective function involves the total cost of opening facilities and serving the clients. The difference is that now the feasible solutions must satisfy the integrality requirement for the variables x_i ($i \in I$) (which corresponds to the condition $S \subseteq I$) and conform to the auxiliary optimization problem. The first constraint in the latter problem requires assigning a supplier to each client. The second constraint allows choosing suppliers only among the open facilities. The vector x_i is assumed to be given.

There are several techniques for reducing this two-level problem to an integer linear program (see [11, 12]). Note that, for each $j \in J$, only the ordering of the elements g_{ij} is important but not their values. Let us arrange the elements in the jth column in ascending order $g_{i_1 j} < \ldots < g_{i_m j}$ and set $S_{ij} = \{l \in I | g_{lj} < g_{ij}\}$ ($i \in I$). The optimal solution $x_{ij}^*(x_i)$ of the client problem has the following property: $x_{ij}^* = 1 \Rightarrow x_l = 0$ for $l \in S_{ij}$. Using this relation, the original two-level problem can be written in the form (see [11, 12]): find

$$\min \sum_{i \in I} f_i x_i + \sum_{j \in J} \sum_{i \in I} c_{ij} x_{ij}$$

subject to

$$x_{ij} + x_l \le 1, \quad l \in S_{ij}, \ i \in I,$$

$$\sum_{i \in I} x_{ij} = 1, \quad 0 \le x_{ij} \le x_i, \quad x_i, x_{ij} \in \{0, 1\}, \quad i \in I, \quad j \in J.$$

The first inequality guarantees that x_{ij} is optimal in the client problem. If the matrices (c_{ij}) and (g_{ij}) are identical, we obtain the simple location problem.

The first inequality can be written in the equivalent form ($i \in I, j \in J$)

$$\sum_{l \in S_{ij}} x_l \le |S_{ij}|(1 - x_{ij}),$$

or

$$x_i \le x_{ij} + \sum_{l \in S_{ij}} x_l,$$

or

$$x_i \le x_{ij} + \sum_{l \in S_{ij}} x_{lj}.$$

It can be shown that the last inequality gives better linear relaxation than the three others (see [11]).

It is known (see [12]) that the location problem with clients' preferences can be reduced to the minimization problem for a pseudo-Boolean function. Let the permutation $i_1^j, \ldots, i_m^j$ specify an ordering of the elements of the jth column of the matrix (g_{ij}). For $j \in J$, we set

$$\nabla c_{i_1 j} = c_{i_1 j},$$

$$\nabla c_{i_l j} = c_{i_l j} - c_{i_{l-1} j}, \quad 1 < l \le m,$$

and define the pseudo-Boolean function

$$B(z) = \sum_{i \in I} f_i(1 - z_i) + \sum_{j \in J} \sum_{i \in I} \nabla c_{ij} \prod_{l \in S_{ij}} z_l.$$

The location problem with clients' preferences is equivalent to the minimization problem for $B(z)$, where $z \ne (1, \ldots, 1)$. The optimal solutions z^* and x^* of these problems are related by the equation $z_i^* = 1 - x_i^*$ $(i \in I)$, and the values of the objective functions on these solutions are identical.

Note that the coefficients ∇c_{ij} can have an arbitrary sign. In other words, for any pseudo-Boolean function, there exists an equivalent location problem with clients' preferences and conversely. Moreover, by analogy with Theorem 2, for any location problem with clients' preferences, the initial data can be restructured so as to minimize the number of clients in a time that polynomially depends on n and m.

Theorem 3. *For an arbitrary pseudo-Boolean function, an equivalent location problem with clients' preferences with the minimal number of clients can be found in a time that polynomially depends on n and m.*

3. ANTAGONISTIC LOCATIONS

Consider the case when two companies sequentially make decisions concerning the location of their facilities. The first company (called leader) enters the market by opening the set of facilities $S_0 \subset I$. Knowing this decision, the rival firm opens the facilities $S_1 \subset I$ $(S_0 \cap S_1 = \varnothing)$. Every client chooses a most suitable facility from the set $S_0 \cup S_1$. Suppose that the service of the client j brings the profit $r_j > 0$, the leader opens p^0 facilities, and the follower opens p^1 facilities. Then, depending on the location of these facilities, the market (the set of clients) is divided between the two companies. Each company wants to maximize its own market share. Thus, we have a two-person antagonistic game. The players are not equal. The leader is the first to make a decision. It can open its facilities at any place. Then, the follower makes a decision knowing the decision of the leader. We have a Stackelberg game in which the market share (the total profit) of the first player is to be maximized. Define the variables of the game.

The leader:

$$x_i = \begin{cases} 1, & \text{if the leader opens a facility at location } i, \\ 0, & \text{otherwise;} \end{cases}$$

the follower:

$$y_i = \begin{cases} 1, & \text{if the follower opens a facility at location } i, \\ 0, & \text{otherwise;} \end{cases}$$

the clients:

$$u_j = \begin{cases} 1, & \text{if client } j \text{ is served by a leader's facility,} \\ 0, & \text{if client } j \text{ is served by a follower's facility.} \end{cases}$$

For the given vector $x_i \in \{0, 1\}$ ($i \in I$), define the set

$$I_j(x) = \{i \in I | g_{ij} < \min_{l \in I}(g_{lj} | x_l = 1)\}, \quad j \in J.$$

This set specifies the sites for the facilities that enable the follower seize the client j. Using the variables defined above, the corresponding two-level program is written as follows: find

$$\max_{x_i} \sum_{j \in J} r_j u_j^*(x_i)$$

subject to

$$\sum_{i \in I} x_i = p^0, \quad x_i \in \{0, 1\}, \quad i \in I,$$

where $u_j^*(x)$, $y_i^*(x)$ is the optimal solution of the follower problem

$$\max_{y_i, u_j} \sum_{j \in J} r_j(1 - u_j)$$

subject to

$$\sum_{i \in I} y_i = p^1,$$

$$1 - u_j \le \sum_{i \in I_j(x)} y_i, \quad y_i + x_i \le 1, \quad y_i, u_j \in \{0, 1\}, \quad i \in I, \quad j \in J.$$

The total profit of the leader is used as the objective function. The set of feasible solutions is described using the follower problem. The vector x_i ($i \in I$) and the sets $I_j(x)$ ($j \in J$) in the follower problem are assumed to be known. No efficient methods for solving this problem are presently known. The first steps in this direction were made in [13], where an exact implicit enumeration method was proposed. In [14], a particular case in which the follower opens a single facility is examined.

In the follower problem, we assumed that it does not open its facilities at the locations where there is a facility of the leader—this is the condition $S_0 \cap S_1 = \emptyset$. This constraint may be removed. If the constraint $y_i + x_i \le 1$ ($i \in I$) is removed from the follower problem, its optimal solution does not change. It is unfavorable for the follower to open its facilities at the sites where there is a leader's facility. However, this situation can change if we set $I_j(x) = \{i \in I | g_{ij} \le \min_{l \in I}(g_{lj} | x_l = 1)\}$ for $j \in J$. In this case, the leader looses a client if the follower opens an equally profitable facility as the leader. Here, we have the model with *inquisitive* clients that are attracted by a new facility other conditions being equal. A survey of various strategies of the client behavior in competitive location models can be found in [15].

Let us show how the two-level programming problem stated above relates to pseudo-Boolean functions. Note that $u_j^* = \prod_{i \in I_j(x)} (1 - y_i^*)$. Then, the problem can be written as follows: find

$$\max \sum_{j \in J} r_j \prod_{i \in I_j(x)} (1 - y_i^*(x_i))$$

subject to

$$\sum_{i \in I} x_i = p^0, \quad x_i \in \{0, 1\}, \quad i \in I,$$

where $y_i^*(x_i)$ is the optimal solution of the follower problem

$$\max \sum_{j \in J} r_j \left(1 - \prod_{i \in I_j(x)} (1 - y_i)\right)$$

subject to

$$\sum_{i \in I} y_i = p^1, \quad y_i \in \{0, 1\}, \quad i \in I.$$

Eliminating the constant from the follower's objective function, we obtain a two-level program for the pseudo-Boolean function $\Sigma_{j \in J} r_j \Pi_{i \in I_j(x)} (1 - y_i)$.

Let us show how the integer linear programming techniques can be used to obtain a family of upper bounds on the optimal value of the leader's objective function and, respectively, a family of lower bounds on the optimal value of the follower's objective function. The main idea underlying the construction of such bounds is as follows. We add an additional constraint to the follower problem. As a result, the optimal value in the follower problem cannot increase, and the optimal value in the leader problem cannot decrease. If the additional constraint allows us to reduce the original two-level program to an integer linear program, we obtain the desired bound. The problem is to find a constraint that possesses the desired properties.

Assume that the follower is guided by the following rule when choosing the sites for its facilities. It orders the possible sites of facilities using a certain criterion. The ordering is performed before solving the problem and is brought to the leader's notice. As soon as the leader announces its decision, the follower opens its facilities at the sites that were not occupied by the leader using the order established earlier. This strategy does not guarantee the optimal solution for the follower, and, therefore, yields a lower bound on its optimum. This bound depends on the ordering. Using various orderings, we obtain different lower bounds. Therefore, we obtain a family of m! lower bounds for the follower or upper bounds on the leader's optimal value.

Now, we demonstrate how the problem subject to the constraint specified above can be reduced to an integer linear program. Without loss of generality, we assume that the possible facility locations are already ordered according to the follower's criterion. The first facility $i = 1$ is the most preferable for the follower and the last facility $i = m$ is the less preferable. Then, for the chosen x_i ($i \in I$), the follower's behavior is unambiguously determined by the following system of constraints:

$$x_i + y_i \le 1, \quad i \in I,$$

$$\sum_{i \in I} y_i = p^1,$$

$$x_i + y_i \ge y_k, \quad i, k \in I, \quad k > i.$$

The last inequality forbids the follower to open a facility at the location k if no facilities were opened by the leader or by the follower at the sites with smaller indexes. Define the additional variables

$$z_{jk} = \begin{cases} 1, & \text{if client } j \text{ is served from location } k, \\ 0, & \text{otherwise.} \end{cases}$$

Given x_i and y_i, the variables z_{jk} are uniquely determined using the following system of constraints:

$$\sum_{k \in I} z_{jk} = 1,$$

$$z_{jk} \le x_k + y_k,$$

$$z_{jk} + \sum_{l \in S_{kj}} z_{jl} \ge x_k + y_k, \quad k \in I, \quad j \in J.$$

The first equality requires that each client be served by a leader's or follower's facility. The second constraint allows the clients to be served only by open facilities. The last inequality establishes a priority in choosing the facilities for serving the clients. Namely, the client is served by the most preferable facility among the leader's and the follower's facilities. Moreover, this constraint is stronger than the first constraint in the preceding system because $z_{jk} + \Sigma_{l \in S_{kj}} z_{jl}$ is not greater than unity.

For the given z_{jk}, x_i and y_i, the variables u_j are uniquely determined by the following system of constraints:

$$u_j \ge z_{jk} - y_k, \quad 1 - u_j \ge z_{jk} - x_k, \quad j \in J, \quad k \in I.$$

The first inequality requires that the client j be served by the leader if $z_{jk} = 1$ and $y_k = 0$. If $z_{jk} = 1$ and $y_k = 1$, the second inequality requires that this client be served by the follower.

Therefore, after x_i ($i \in I$) are chosen, y_i ($i \in I$), z_{jk} ($j \in J$, $k \in I$), and u_j ($j \in J$) are determined uniquely. The optimal values of all the variables are determined by the following problem: find

$$\max \sum_{j \in J} r_j u_j$$

subject to

$$\sum_{i \in I} x_i = p^0, \quad \sum_{i \in I} y_i = p^1,$$

$$x_i + y_i \ge y_k, \quad k > i, \ k, i \in I$$

$$\sum_{k \in I} z_{jk} = 1, \ j \in J$$

$$z_{jk} \le x_k + y_k, \ j \in J, k \in I$$

$$z_{jk} + \sum_{l \in S_{kj}} z_{jl} \ge x_k + y_k, \ j \in J, k \in I$$

$$u_j \ge z_{jk} - y_k, \quad 1 - u_j \ge z_{jk} - x_k, \ j \in J, k \in I$$

$$x_i, y_i, z_{jk}, u_j \in \{0, 1\}, \quad k, i \in I, \quad j \in J.$$

Let $(x_i^*, y_i^*, z_{jk}^*, u_j^*)$ be the optimal solution of this problem. It yields an upper bound on the leader's optimum. A complete or partial linear relaxation of this problem also yields an upper bound. To obtain a lower bound, it is sufficient to solve the follower problem for the given x_i^*. Other approaches to finding lower bounds and a comparison of those approaches can be found in [16, 17]. Thus, we obtain an approximate solution of the original two-level program with an a posteriori estimate of accuracy.

4. A GAME MODEL OF FACILITY LOCATION

Consider the situation when p companies simultaneously open their facilities for serving clients. The goal of each company (player) is to maximize its own profit. We assume that the players make decisions independently of each other, do not form coalitions, and pursue only their selfish goals. For the simplicity of the presentation, we assume that every player opens no more than one facility although the reasoning below can be extended for the case of an arbitrary number of facilities.

Let r_j be the maximum price the client j agrees to pay for the product. If the players wanted to cooperate to get the maximum profit, their optimal strategy could be found by solving the problem

$$\max \sum_{j \in J} \sum_{i \in I} (r_j - c_{ij}) x_{ij}$$

subject to

$$\sum_{i \in I} x_{ij} \le 1, \quad x_i \ge x_{ij}, \quad \sum_{i \in I} x_i = p,$$
$$x_i, x_{ij} \in \{0, 1\}, \quad i \in I, \quad j \in J.$$

If the first inequality turns into an equality for all $j \in J$, then each client pays the maximum admissible price, and the entire profit goes to the players.

Consider the situation when the players act independently of each other. Each of them can open a facility at any location belonging to the set I. The expenses of the kth player for producing and delivering the products are given by the matrix c_{ij}^k $(j \in J)$. Without loss of generality, we may assume that $c_{ij}^k \le r_j$ for $j \in J$. Otherwise, we set $c_{ij}^k = r_j$, and the player gets no profit for servicing this client.

Suppose that the players opened the facilities $i_1, \ldots, i_p$. What is the price q_j of their products for the client j? Denote by c_j the minimal total cost of production and delivery for servicing this client by the open facilities:

$$c_j = \min\{c_{i_1 j}^1, \ldots, c_{i_p j}^p\}, \quad j \in J.$$

The price q_j cannot be less than c_j because it is unprofitable to supply the goods at such a low price. Assume that all the players offered the price $q_j = r_j$. Then, it is of no importance for the client which vendor supplies the products. Since every player wants to be a supplier, they start reducing their prices. Let $i(j) = \operatorname{argmin}\{c_{i_1 j}^1, \ldots, c_{i_p j}^p\}$ and c_j' be the second minimal element among $c_{i_1 j}^1, \ldots, c_{i_p j}^p$. The price reduction will stop at c_j' when the player $i(j)$ becomes the only one for which the service of the client j is still profitable. It is little sense in reducing the price further. On the other hand, it is unreasonable to raise the price because a competitor can appear for which the service of this client is also profitable. Therefore, we have $q_j = c_j'$ for $j \in J$. If $c_j = c_j'$ for a certain $j \in J$, then $q_j = c_j$, and none of the players gains profit by servicing this client.

When the players had the common interests, they maintained the price q_j at the level r_j and grabbed the entire profit. Now, $q_j = c_j'$, and the profit $r_j - c_j$ is divided between the supplier $i(j)$, which gains $q_j - c_j$, and the client, which saves $r_j - q_j$. Denote by Γ_k the set of clients served by the kth player for the given choice $i_1, \ldots, i_p$. Then, the profit of the kth player is

$$w_k = \sum_{j \in \Gamma_k} (q_j - c_j),$$

the total savings of the clients is

$$\nu(i_1, \ldots, i_p) = \sum_{j \in J} (r_j - q_j),$$

and the total profit of the players and savings of the clients is

$$\mu(i_1, \ldots, i_p) = \sum_{j \in J} (r_j - c_j) = \sum_{k=1}^{p} w_k + \nu(i_1, \ldots, i_p).$$

The solution $(i_1, \ldots, i_p)$ is called a Nash equilibrium or an equilibrium solution if none of the players can increase its own profit when the other players do not change their choices. A solution is said to be optimal if $\mu(i_1, \ldots, i_p)$ has the maximum possible value. Indeed, this solution is the best one for the community because it gives the maximum effect from the facility location and product manufacturing. Below, we show that any optimal solution is an equilibrium one but not every equilibrium solution is optimal. In [18], the concept of the price of anarchy is introduced. This is the ratio of the optimal solution to the worst solution among the Nash equilibriums. It is known (see [19]) that the price of anarchy in this game does not exceed two. However, the problem of the complexity of finding equilibrium solutions remains open. It is not clear whether at least one equilibrium solution can be found in a time that polynomially depends on the length of the initial data.

Consider the optimization problem

$$\min\left(\sum_{j\in J}\sum_{k=1}^{p}\sum_{i\in I}c_{ij}^{k}x_{ij}^{k}\right)$$

subject to

$$\sum_{k=1}^{p}\sum_{i\in I}x_{ij}^{k} = 1, \quad j\in J,$$

$$x_i^k \ge x_{ij}^k, \quad i\in I, \quad j\in J, \quad k = 1, 2, \dots, p,$$

$$\sum_{i\in I}x_i^k = 1, \quad k = 1, 2, \dots, p,$$

where the variables x_i^k and x_{ij}^k have the following meaning:

$$x_i^k = \begin{cases} 1, \text{ if the player } k \text{ opens a facility at the location } i; \\ 0, \text{ otherwise,} \end{cases}$$

$$x_{ij}^k = \begin{cases} 1, \text{ if the client } j \text{ is served by the player } k \text{ from the facility } i; \\ 0, \text{ otherwise.} \end{cases}$$

The objective function of this problem is interpreted as the production and delivery costs of the players. The first constraint requires all the clients to be served. The second constraint allows the clients to be served only by open facilities. The last constraint allows each player to open only one facility.

Note that, in this problem, several players may open their facilities at the same place. For that reason, this statement is close to the p-median problem (see [20]) but not identical to it. We call it the *p-median game* or PMG. Let (x_i^k, x_{ij}^k) be a feasible solution of this problem and, for the given x_i^k, the quantities x_{ij}^k determine the optimal assignment of the clients to the open facilities. In other words, we assume that x_{ij}^k are determined by the variables x_i^k, and the solution can be specified using only these variables. Then, the problem can be written as follows: find

$$\min\sum_{j\in J}\min_{k=1,2,\dots,p}\min_{i\in I}\{c_{ij}^k \mid x_i^k = 1\}$$

subject to

$$\sum_{i\in I}x_i^k = 1, \quad k = 1, 2, \dots, p,$$

$$x_i^k \in \{0, 1\}, \quad k = 1, 2, \dots, p, \quad i\in I.$$

Let (x_i^k) be a feasible solution of the problem. By the neighborhood Swap(x), we mean the set of the feasible solutions that can be obtained from (x_i^k) by choosing a certain k and replacing the facility of this player by any other facility. A local minimum for this neighborhood is defined as a solution for which the value of the objective function does not exceed the value of the objective function on any neighboring solution.

Lemma 2 (see [21, 22])**.** *There is a one-to-one correspondence between the local minima of the PMG and the equilibrium solutions.*

Proof. Let $(i_1, \ldots, i_p)$ be a solution. We associate with it the solution (x_i^k) defined as follows: $x_i^k = 1$ if $i = i_k$ and $x_i^k = 0$, otherwise. We show that this solution is a local minimum if and only if $(i_1, \ldots, i_p)$ is an equilibrium solution. To verify this fact, it is sufficient to show that, when the kth player relocates its facility, the change in the player's profit is equal to the change of the objective function in the PMG when the solution (x_i^k) is replaced by the corresponding neighboring solution.

Suppose that the player k moved its facility i_k to the site l and the new set of its clients $\tilde{\Gamma}_k$ does not overlap with the set Γ_k. For $j \in \tilde{\Gamma}_k$, the profit of this player gained by servicing this client is $c_j - c_{lj}^k \geq 0$. This is identical to the change of the jth term of the objective function of the PMG when a solution is changed for an adjacent one:

$$\tilde{x}_i^t = \begin{cases} 1, & i = l, \\ 0, & i \neq l, \end{cases} \quad t = k, \quad \text{and} \quad \tilde{x}_i^t = x_i^t, \quad t \neq k.$$

For $j \in \Gamma_k$, the profit of the player was $q_j - c_j \geq 0$; now, it looses this profit. This is exactly equal to the increase of the jth term of the objective function of the PMG. Therefore, the relocation of its facility by the kth player results in the change of its own profit from $\sum_{j \in \Gamma_k}(q_j - c_j)$ to $\sum_{j \in \tilde{\Gamma}_k}(c_j - c_{lj}^k)$, and this is exactly equal to the change in the objective function of the PMG. This completes the proof of Lemma 2.

Therefore, we see that finding equilibrium solutions is closely related with the problem of finding local minima. We consider this problem in more detail. First, recall the main definitions (see [23, 24]).

Definition 1. *An optimization problem* (OP) is defined by the following set of objects: $\langle \mathcal{I}, \mathrm{Sol}, F, \mathrm{goal} \rangle$, where $\mathcal{I}$ is the set of inputs, Sol is the function that assigns a set of feasible solutions $\mathrm{Sol}(x)$ to each input $x \in \mathcal{I}$, F is the function that determines the weight $F(s, x)$ of the feasible solution s for the input x, goal $\in$ {min, max} indicates whether the problem is a maximization or a minimization one.

In the optimization problem, we want to find an optimal solution for the given input x.

Below, we consider only minimization problems.

Definition 2. A *local search problem* is defined by the pair $\Pi = (\mathrm{OP}, N)$, where OP is an optimization problem and N is a neighborhood function that assigns a set $N(x, s) \subseteq \mathrm{Sol}(x)$ of neighboring solutions to every feasible solution s for the input x. The local search problem is to find a local minimum for the given input x.

Definition 3. The local search problem Π belongs to the class PLS if there are three polynomial-time algorithms A, B, and C, and a polynomial q such that the following conditions are satisfied.

1. The algorithm A determines if an arbitrary given string x is an input of the problem. If $x \in \mathcal{I}$, then the algorithm finds a feasible solution of the OP.

2. For any input $x \in \mathcal{I}$ of the problem and any string s, the algorithm B determines if s is a feasible solution. If $s \in \mathrm{Sol}(x)$, then this algorithm finds the value of the objective function $F(s, x)$ in polynomial time.

3. For any input $x \in \mathcal{I}$ and any solution $s \in \mathrm{Sol}(x)$, the algorithm C determines if s is a local minimum. If it is not, the algorithm finds a neighboring solution $s' \in N(x, s)$ with a lower value of the objective function.

4. For any input $x \in \mathcal{I}$, the length of any feasible solution $s \in \mathrm{Sol}(x)$ is polynomially bounded by the length of the problem's input; that is, $|s| \leq q(|x|)$.

It is easy to verify that the local search problem for a PMG with the neighborhood Swap belongs to the class PLS.

Definition 4. Let Π_1 and Π_2 be two local search problems. The problem Π_1 is PLS-reducible to the problem Π_2 if there exist two polynomial-time computable functions h and g such that

1. given an arbitrary input x of Π_1, the function h produces an input $h(x)$ of Π_2;

2. given an arbitrary solution s for the input $h(x)$, the function g produces a solution $g(s, x)$ for the input x;

3. if $x \in \Pi_1$ and s is a local minimum for the input $h(x) \in \Pi_2$, then $g(s, x)$ is a local minimum for the input x.

The problem Π in the class PLS is said to be *PLS-complete* if any problem in this class can be reduced to it. Examples of PLS-complete problems can be found in [24]. In this paper, we show that the local minimization for the PMG with the neighborhood Swap is a PLS-complete problem; i.e., it is the most difficult problem in this class. Moreover, we obtain an exponential lower bound on the number of steps in the worst case for the local improvement algorithms. This bound is independent of the pivoting rule; i.e., it is independent of the method used to find the best neighboring solution in the neighborhood; however, it does not exclude the possibility to find a local optimum (Nash equilibrium) in polynomial time using other algorithms.

Definition 5. *The transition graph* $TG_\Pi(x)$ for the input x of the problem Π is defined as the directed graph with one vertex for each feasible solution of the problem. Two nodes are connected by an arc (s, s') if s' is a neighboring solution for s and $F(s, x) > F(s', x)$. The height of the vertex s is the length of the shortest path in the graph $TG_\Pi(x)$ from the vertex to the sink, that is, to the local minimum. The height of $TG_\Pi(x)$ is the maximum height of its vertices.

The height of a vertex is essentially a lower bound on the number of steps of the local improvement algorithm independent of the pivoting rule. Therefore, an algorithm requires an exponential number of steps if and only if there are the initial data of the problem for which the height of the transition graph is an exponential function of the length of the initial data.

Definition 6. Suppose that the problem Π_1 is PLS reducible to the problem Π_2 and h and g are the corresponding functions. The reducibility is said to be *tight* if, for any input x of Π_1, there exists a subset R of feasible solutions for the input $y = h(x)$ of Π_2 such that the following conditions are satisfied:

1. R contains all the local minima for the input y.

2. There exists a polynomial-time algorithm that, for each solution p for the input x, finds a solution $q \in R$ for the input y such that $g(q, x) = p$.

3. Let the transition graph $TG_{\Pi_2}(y)$ contain a directed path from the vertex $q \in R$ to the vertex $q' \in R$ such that it does not contain intermediate vertices from R, and let $p = g(q, x)$ and $p' = g(q', x)$ be the corresponding solutions for the input x. Then, $p = p'$ or the transition graph $TG_{\Pi_1}(x)$ contains the arc leading from p to p'.

The problem Π from the class PLS is said to be tightly complete if all the problems in this class are tightly reducible to it. It is known (see [25]) that the following problem on coloring the graph vertices in two colors is tightly complete. Let an undirected graph $G = (V, E)$ with the weights w_e, $e \in E$ on its edges be given. Any coloring of the graph vertices in two colors is a feasible solution of this problem; this is a function $c : V \longrightarrow \{0, 1\}$. The total weight of the edges of the same color is the objective function. The neighborhood *Flip* consists of all the colorings that differ from the given coloring in the color of exactly one vertex. The problem is to find a coloring that provides a local minimum in the neighborhood *Flip.*

Theorem 4. *The* Flip-*minimal coloring problem tightly reduces to the local search problem* (*PMG*, *Swap*).

Proof. Given the graph, we construct the initial data of the PMG problem. Set $I = \{1, 2, \ldots, 2|V|\}$, $J = \{1, 2, \ldots, |V| + 2|E|\}$, $p = |V|$, and $W = \sum_{e \in E} |w_e| + 1$. To every player, we assign the same matrix c_{ij} as follows.

To each vertex $v \in V$, we assign two rows i_v and i'_v and a column j_v. To each edge $e = (j_1, j_2) \in E$, we assign two columns $j_1(e)$ and $j_2(e) \in J$. For $j_v = 1, 2, \ldots, |V|$, we set

$$c_{ij_v} = \begin{cases} 0 & \text{if} \quad (i = i_v) \vee (i = i'_v), \\ W, & \text{otherwise}. \end{cases}$$

For $e = (j_1, j_2) \in E$ and $w_e \geq 0$, we set

$$c_{ij_1(e)} = \begin{cases} 0 & \text{if} \quad (i = j_1(e)) \vee (i = j_2(e)), \\ w_e & \text{if} \quad (i = j'_1(e)) \vee (i = j'_2(e)), \\ W, & \text{otherwise}, \end{cases}$$

$$c_{ij_2(e)} = \begin{cases} 0, & \text{if} \quad (i = j'_1(e)) \vee (i = j'_2(e)), \\ w_e, & \text{if} \quad (i = j_1(e)) \vee (i = j_2(e)), \\ W, & \text{otherwise}. \end{cases}$$

If $w_e < 0$, we set

$$c_{ij_1(e)} = \begin{cases} w_e/2, & \text{if} \quad (i = j_1(e)) \vee (i = j'_2(e)), \\ -w_e/2, & \text{if} \quad (i = j'_1(e)) \vee (i = j_2(e)), \\ W, & \text{otherwise}, \end{cases}$$

$$c_{ij_2(e)} = \begin{cases} w_e/2, & \text{if} \quad (i = j'_1(e)) \vee (i = j_2(e)), \\ -w_e/2, & \text{if} \quad (i = j_1(e)) \vee (i = j'_2(e)), \\ W, & \text{otherwise}. \end{cases}$$

The structure of the matrix (c_{ij}) is as follows:

$$\begin{array}{c} \\ \\ i_v \\ \\ \\ \\ \\ i'_v \\ \\ \\ \end{array}
\left[\begin{array}{ccc|cc|cc}
 & j_v & & j_1(e) & j_2(e) & j_1(e) & j_2(e) \\
0 & \vdots & & \vdots & \vdots & 0.5w_e & -0.5w_e \\
 & \ddots & & \vdots & \vdots & -0.5w_e & 0.5w_e \\
\cdots & 0 & & \vdots & \vdots & \vdots & \vdots \\
 & \vdots & \ddots & 0 & w_e & \vdots & \vdots \\
 & \vdots & 0 & 0 & w_e & \vdots & \vdots \\
\hline
0 & \vdots & & \vdots & \vdots & -0.5w_e & 0.5w_e \\
 & \ddots & & \vdots & \vdots & 0.5w_e & -0.5w_e \\
\cdots & 0 & & \vdots & \vdots & \vdots & \vdots \\
 & & \ddots & w_e & 0 & \vdots & \vdots \\
 & & 0 & w_e & 0 & \vdots & \vdots \\
\end{array}\right]$$

$$\qquad\qquad w_e \ge 0 \qquad\qquad w_e < 0$$

The proposed structure defines a polynomially computable function h in Definition 4. The function g is defined as follows. Let (x_i^k) be a feasible solution of the PMG problem. Set

$$c_x(v) = \begin{cases} 0 & \text{if there exists a } k, \text{ such that } x_{i_v}^k = 1, \\ 1, & \text{otherwise}. \end{cases}$$

Let us verify that $c_x(v)$ is a local minimum with respect to the neighborhood *Flip* if and only if (x_i^k) is a local minimum for Swap. Note that any local minimum (x_i^k) possesses the following property: for any $v \in V$, there exists a unique number k such that $x_{i_v}^k + x_{i'_v}^k = 1$. First, we show that there are no vertices $v \in V$ such that $x_{i_v}^{k_1} + x_{i_v}^{k_2} = 1$ and $k_1 \neq k_2$. Indeed, assume that, for a certain $v \in V$, there are such indexes. Since $p = |V|$, there exists a vertex $v_0 \in V$ such that $x_{i_{v_0}}^t = 0$ for all $t = 1, 2, \ldots, p$. Construct a new solution $\bar{x}$ that differs from the preceding solution only in the components (k_1, i_{v_0}) and (k_1, i_v). Set $\bar{x}_{i_{v_0}}^{k_1} = 1$ and $\bar{x}_{i_v}^{k_1} = 0$. This solution is a neighbor of the original one, and it has a smaller value of the objective function. Indeed, since $x_{i_{v_0}}^{k_1} =$

$x^{k_1}_{i'_{v_0}} = 0$, the term with the index j_{v_0} in the objective function of the PMG equals W. Setting $\bar{x}^{k_1}_{i_{v_0}} = 1$, we obtain a decrease of the objective function by W. The change of $x^{k_1}_{i_v} = 1$ by $\bar{x}^{k_1}_{i_v} = 0$ cannot lead to the increase of the objective function because $x^{k_2}_{i_v} = 1$. The cases $x^{k_1}_{i'_v} = x^{k_2}_{i_v} = 1$ and $x^{k_1}_{i'_v} = x^{k_2}_{i'_v} = 1$ are analyzed similarly.

Therefore, if (x^k_i) is a local minimum for the neighborhood Swap, there is a unique index $k(v)$ for each vertex $v \in V$ such that $x^{k(v)}_{i_v} + x^{k(v)}_{i'_v} = 1$. We show that $c_x(v)$ is a local minimum for the neighborhood Flip. Assume that the edge $e \in E$ is incident to some vertices of the same color. Consider two cases.

1. Let $w_e \ge 0$. For the edge $e = (j_1(e), j_2(e))$, find the rows i_1 and i_2 in the matrix (c^k_{ij}) corresponding to the vertices $j_1(e)$ and $j_2(e)$. Let $k_1 = k(j_1(e))$ and $k_2 = k(j_2(e))$. If $x^{k_1}_{i_1} = x^{k_2}_{i_2} = 1$, then $x^{k_1}_{i'_1} = x^{k_2}_{i'_2} = 0$, and the term for $j = j_1(e)$ in the objective function vanishes; the term for $j = j_2(e)$ is w_e; that is, the weight w_e is taken into account both in the coloring problem and in the PMG. If $x^{k_1}_{i_1} = x^{k_2}_{i_2} = 0$, then $x^{k_1}_{i'_1} = x^{k_2}_{i'_2} = 1$, and the term for $j = j_1(e)$ in the objective function is w_e; the term for $j = j_2(e)$ vanishes. Therefore, we arrive at the same conclusion.

2. Let $w_e < 0$. If $x^{k_1}_{i_1} = x^{k_2}_{i_2} = 1$, then both terms in the objective function are $w_e/2$, which sums to w_e. If $x^{k_1}_{i'_1} = x^{k_2}_{i'_2} = 1$, we again have $w_e/2 + w_e/2 = w_e$.

Now, consider an edge connecting two differently colored vertices. Let $x^{k_1}_{i_1} = 1$ and $x^{k_2}_{i_2} = 0$. If $w_e \ge 0$, then the terms for $j = j_1(e)$ and $j = j_2(e)$ vanish, and the weight of this edge is not included in the objective function. If $w_e < 0$, then the term for $j = j_1(e)$ is $w_e/2$ and the term for $j = j_2(e)$ is $-w_e/2$, which sums to zero. Therefore, the values of the objective functions on the solutions (x^k_i) and $c_x(v)$ are identical.

Assume that the solution $c_x(v)$ is not a local minimum. Then, there exists a vertex $v \in V$ such that the change of its color results in the decrease of the objective function. In this case, the solution (x^k_i) is a not local minimum for the neighborhood Swap because the change of the color of the vertex v corresponds to the change of variables

$$x^k_{i_v} := 1 - x^k_{i_v}, \quad x^k_{i'_v} := 1 - x^k_{i'_v} \quad \text{for} \quad k = k(i_v)$$

with the same change in the objective function. Therefore, $c_x(v)$ is a local minimum, and the reduction described above satisfies Definition 4.

Let us verify that this reduction is tight. As the set R, we use all the feasible solutions (x^k_i) possessing the property $x^{k(v)}_{i_v} + x^{k(v)}_{i'_v} = 1$ for any $v \in V$. We have already proved that this set contains all the local minima of the PMG; therefore, Condition 1 in Definition 6 is satisfied. Let us verify Condition 2. Let $c(v)$ be an arbitrary coloring of the vertices in two colors. For the kth player, find a vertex v such that $i_v = k$ and set

$$x^k_{i_v} = \begin{cases} 1 \text{ if the vertex } v \text{ has color } 0, \\ 0, \text{ otherwise}, \end{cases}$$

$$x^k_{i'_v} = \begin{cases} 0 \quad \text{if the vertex } v \text{ has color } 0, \\ 1, \text{ otherwise}. \end{cases}$$

For $k \ne i_v$, set $x^k_{i_v} = x^k_{i'_v} = 0$. We obtain a one-to-one correspondence between the colorings and the elements of the set R.

Let us check the last condition. Note that the transition graph of the PMG does not contain arcs of the form (q_1, q_2), where $q_1 \in R$ and $q_2 \notin R$, because the value of the objective function for q_2 is always greater than that for q_1. In other words, any path that begins and ends in R entirely belongs to R. A path without intermediate vertices in R can only be an arc of the form (q_1, q_2), where $q_1, q_2 \in R$. However, such an arc has a corresponding arc in the transition graph of the coloring problem. This completes the proof of the theorem.

Consider the complexity of obtaining equilibrium solutions using the following iterative algorithm. Let (x_i^k) be not an equilibrium solution; that is, there is a player (or several players) that can increase its own profit by relocating its facility. One step of this algorithm consists in finding such a player and choosing a new location for its facility that increases its own profit. The procedure of selecting a new location can be arbitrary. Our aim is to estimate the number of steps of such an iterative algorithm in the worst case for any selection procedure.

Corollary 1. In the worst case, the iterative algorithm requires an exponential number of steps for finding an equilibrium solution with any procedure used to select a player and a better facility for it.

The validity of this corollary follows from the tight PLS completeness of the local search problem (PMG, Swap) and from the existence of an exponentially high problem in the class PLS (see [24]). The estimate remains exponential when any rule for selecting a player and a better facility for it (deterministic, probabilistic, or any other arbitrarily complex rule) is used.

In the problem of finding equilibrium solutions, we want to find at least one (any) equilibrium solution. However, if the players' decisions are already known and they are ready to change them only if the profit increases, another, more practical problem arises: find an equilibrium solution that can be obtained by an iterative local improvement algorithm from the given initial position. In other words, a starting vertex is given in the transition graph. It is required to find a sink that is reachable from the starting vertex using a directed path. It turns out that this problem is much more difficult than the preceding one.

Corollary 2. Finding an equilibrium solution for the given initial position of the players is a PSPACE-complete problem.

Proof. We prove that the problem under examination belongs to the class PSPACE. It follows from the definition of the class PLS that the length of any solution is bounded by a polynomial the length of the input. Since the iterative algorithm does not require that all the intermediate players' positions be stored and the best neighboring solution can be found in a polynomial time, the required amount of memory is also bounded by a polynomial depending on the length of the input data. Therefore, the problem belongs to the class PSPACE.

It is known (see [24]) that the class PLS includes local search problems with a fixed starting point that are PSPACE-complete. The problem of coloring graph vertices with the neighborhood Flip is one of them. The tight reducibility established in Theorem 4 implies the polynomial reducibility of the corresponding problems with fixed initial solutions (see [24]). Therefore, finding an equilibrium solution that is reachable from the given initial position of the players is a PSPACE-complete problem. The theorem is proved.

Now, assume that each player may open several facilities. In this case, an equilibrium solution is defined as a solution in which none of the players can increase its own profit by changing its decision for any other. For this case, it is easy to write the corresponding optimization problem and determine an appropriate neighborhood. The tight completeness of this problem with all the ensuing consequences was proved in [21].

CONCLUSIONS

In this paper, we considered two classes of competitive location models. The first class leads to discrete two-level models for unequal players. Even the construction of a feasible solution in this problem requires that the follower problem be exactly solved, which is known to be the NP-hard maximum coverage problem. A novel method for constructing a family of upper and lower bounds on the optimum for the leader is proposed. This family has an exponential cardinality. It enables one to obtain an approximate solution with an a posteriori estimate of the accuracy for the original two-level problem. The analysis of the quality of such bounds and of the complexity of their derivation requires computational experiments. We believe that this direction of research is extremely important because the entire family cannot be examined and the search for the best element can be as difficult as the original problem.

For the second class of models, the relationship between the equilibrium solutions and the local optima is revealed, and the tight PLS completeness of the problem of finding such solutions is proved. As a consequence, we conclude that the standard local improvement algorithm with an arbitrary pivoting rule requires an exponential number of steps in the worst case. Finding an equilibrium solution for the given initial position of the players turns out to be a PSPACE-complete problem. Nevertheless, question of the complexity of finding equilibrium solutions and the local optima remains open. If one succeeds in proving that no polynomial algorithms for finding equilibrium solutions exist, this will imply that $P \neq NP$ (see [20]).

ACKNOWLEDGMENTS

This work was supported by the Analytical Department Targeted Program no. 2.1.1/3235.

REFERENCES

1. Z. Drezner, K. Klamroth, A. Schobel, and G. Wesolowsky, "The Weber Problem," in *Facility Location: Applications and Theory* (Springer, Berlin, 2004), pp. 1–36.
2. V. L. Beresnev, *Discrete Location Problems and Polynomials of Boolean Variables* (Institut matematiki, SO RAN, Novosibirsk, 2005) [in Russian].
3. Y. Chan, *Location, Transport and Land-Use. Modelling Spatial-Temporal Information* (Springer, Berlin, 2005).
4. V. R. Khachaturov, V. E. Veselovskii, A. V. Zlotov, et al., *Combinatorial Methods and Algorithms for Solving Large-Scale Discrete Optimization Problems* (Nauka, Moscow, 2000) [in Russian].
5. H. A. Eiselt and C.-L. Sandblom, *Decision Analysis, Location Models, and Scheduling Problems* (Springer, Berlin, 2004).
6. *Discrete Location Theory,* Ed. By P. B. Mirchandani, and R. L. Francis (Wiley, Chichester, 1990).
7. V. Beresnev, E. Kh. Gimadi, and V. T. Dement'ev, *Extremal Standardization Problems* (Nauka, Novosibirsk, 1978) [in Russian].
8. B. Korte and J. Vygen, *Combinatorial Optimization: Theory and Algorithms*, 3rd ed. (Springer, Berlin, 2005).
9. A. Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency* (Springer, Berlin, 2003).
10. J. Krarup and P. M. Pruzan, "The Simple Plant Location Problem: Survey and Synthesis," European J. Operat. Res. **12**, 36–81 (1983).
11. E. V. Alekseeva and Yu. A. Kochetov, "Genetic Local Search for the *p*-Median Problem with Client's Preferences," Diskretnyi Analiz Issl. Operatsii, Ser. 2, **14** (1), 3–31 (2007).
12. L. E. Gorbachevskaya, *Polynomially Solvable and NP-Hard Standardization Problems*, Candidate's Dissertation in Mathematics and Physics (IM SO RAN, Novosibirsk, 1998).
13. C. M. C. Rodriquez and J. A. M. Perez, "Multiple Voting Location Problems," European J. Operat. Res. **191**, 437–453 (2008).
14. F. Plastria and L. Vanhaverbeke, "Discrete Models for Competitive Location with Foresight," Comput. Operat. Res. **35**, 683–700 (2008).
15. T. Drezner and H. A. Eiselt, "Consumers in Competitive Location Models," in *Facility Location: Applications and Theory* (Springer, Berlin, 2004), pp. 151–178.
16. E. V. Alekseeva and N. A. Kochetova, "Upper and Lower Bounds for the Competitive *p*-Median Problem," in *Trudy XIV Baikal'skoi mezhdunarodnoi shkoly-seminara "Metody optimizatsii i ikh prilozheniya, Severobaikal'sk," 2008***,** Vol. 1, pp. 563–569 [in Russian].
17. E. V. Alekseeva and A. V. Orlov, "A Genetic Algorithm for the Competitive *p*-Median Problem," in *Trudy XIV Baikal'skoi mezhdunarodnoi shkoly-seminara "Metody optimizatsii i ikh prilozheniya, Severobaikal'sk," 2008*, Vol. 1, pp. 570–576 [in Russian].
18. E. Koutsoupias and C. Papadimitriou, "Worst-Case Equilibria," in *Proc. XVI Ann. Symposium on the Theory and Aspects of Computer Science, Trier, Germany, 1999* pp. 404–413.
19. A. Vetta, "Nash Equilibria in Competitive Societies, with Applications to Facility Location, Traffic Routing, and Auctions," in *Proc. XLIII Ann. IEEE Symposium on the Foundations of Computer Science, Vancouver, Canada, 2002* pp. 416–425.
20. Yu. A. Kochetov, M. G. Pashchenko, and A. V. Plyasunov, "On the Complexity of Local Search in the *p*-Median Problem," Diskretnyi Analiz Issl. Operatsii, Ser. 2, **12** (2), 44–71 (2005).
21. Yu. A. Kochetov, "Nash Equilibria in Game Location Models," in *Trudy XIV Baikal'skoi mezhdunarodnoi shkoly-seminara "Metody optimizatsii i ikh prilozheniya, Severobaikal'sk, 2008*, Vol. 1, pp. 119–127 [in Russian].
22. E. Tardos and T. Wexler, "Network Formation Games and the Potential Function Method" in *Algorithmic Game Theory* (Univ. Press, Cambridge, 2007), pp. 487–516.
23. Yu. A. Kochetov, "Computational Bounds for Local Search in Combinatorial Optimization," Zh. Vychisl. Mat. Mat. Fiz. **48** (5), 747–764 (2008) [Comput. Math. Math. Phys. 747–763 **48**, (2008)].
24. M. Yannakakis, "Computational Complexity," in *Local Search in Combinatorial Optimization* (Wiley, Chichester, 1997), pp. 19–55.
25. T. Vredeveld and J. K. Lenstra, "On Local Search for the Generalized Graph Coloring Problem," Operat. Res. Letts **31** (4), 28–34 (2003).