VNS heuristic for the $(r \mid p)$–centroid problem on the plane

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Abstract

In the $(r \mid p)$-centroid problem, two players, called leader and follower, open facilities to service clients. We assume that clients are identified with their location on the Euclidean plane, and facilities can be opened anywhere in the plane. The leader opens p facilities. Later on, the follower opens r facilities. Each client patronizes the closest facility. Our goal is to find p facilities for the leader to maximize his market share. For this Σ_2^P-hard problem we develop the VNS heuristic, based on the exact approach for the follower problem. We apply the $(r \mid X_{p-1}+1)$-centroid subproblem for finding the best neighboring solution according to the swap neighborhood. It is shown that this subproblem is polynomially solvable for fixed r. Computational experiments for the randomly generated test instances show that the VNS heuristic dominates the previous ones.

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1   Introduction

This paper addresses a Stackelberg facility location game on a two–dimensional Euclidean plane. We assume that the clients demands are concentrated at a finite number of points in the plane. In the first stage of the game, a player, called the leader, opens his own p facilities. At the second stage, another player, called the follower, opens his own r facilities. At the third stage, each client chooses the closest opened facility as a supplier. In case of ties, the leader's facility is preferred. Each player tries to maximize his own market share. The goal of the game is to find p points for the leader facilities to maximize his market share. We assume that the leader knows how many facilities the follower will locate.

Such Stackelberg game was studied by Hakimi [4] for location on a network. Following Hakimi, the leader problem is called a *centroid problem* and the follower problem is called a *medianoid problem*. A comprehensive review of complexity results and properties of the problems can be found in [6].

In [1] the alternating heuristic for the centroid problem is suggested. Two greedy strategies are used for the follower problem. In [2] an improved alternating heuristic is developed. In each iteration of the alternating heuristic, we consider the solution of one player and compute the best answer for another player. The branch and bound method is applied for this end. At the end of the alternating process, the clients are clustered, and an exact polynomial-time algorithm for the $(1 \mid 1)$-centroid problem is applied. In [3] it is shown that the centroid problem is Σ_2^P-hard and the medianoid problem is NP-hard in the strong sense.

In this paper we present the VNS local search heuristic using an exact approach for the follower problem. We consider the $(r \mid X_{p-1} + 1)$-centroid problem where the leader moves exactly one facility. We use this problem in order to find the best neighboring solution in the swap neighborhood. It is shown that this problem is polynomially solvable for fixed r. We solve the medianoid problem for many leader solutions, but the number of such solutions is polynomially bounded. Computational results for randomly generated instances from the benchmark library *Discrete Location Problems* (http:// math.nsc.ru/AP/benchmarks/index.html) show that the new approach dominates the previous heuristics.

2   Mathematical model

Let us consider a two–dimensional Euclidean plane in which n clients are located. We assume that each client j has a positive demand w_j. Let X

be the set of p points where the leader opens his own facilities and let Y be the set of r points where the follower opens his own facilities. The distances from client j to the closest facility of the leader and the closest facility of the follower are denoted as $d(j, X)$ and $d(j, Y)$, respectively. Client j prefers Y over X if $d(j, Y) < d(j, X)$ and prefers X over Y otherwise. By

$$U(Y \prec X) := \{j \mid d(j, Y) < d(j, X)\}$$

we denote the set of clients preferring Y over X. The total demand captured by the follower by locating his facilities at Y while the leader locates his facilities at X is given by

$$W(Y \prec X) := \sum (w_j \mid j \in U(Y \prec X)).$$

For X given, the follower tries to maximize his own market share. The maximal value $W^*(X)$ is defined to be

$$W^*(X) := \max_{Y, |Y|=r} W(Y \prec X).$$

This maximization problem will be called the *follower problem.* The leader tries to minimize the market share of the follower. This minimal value $W^*(X^*)$ is defined to be

$$W^*(X^*) := \min_{X, |X|=p} W^*(X).$$

For the best solution X^* of the leader, his market share is $\sum_{j=1}^{n} w_j - W^*(X^*)$. In the $(r \mid p)$–*centroid problem,* the goal is to find X^* and $W^*(X^*)$.

3 The follower problem

Let us first describe an exact approach for the follower problem. Such problem is rewritten as an integer linear programming problem, and solved using a branch and bound method.

For each client j, we introduce a disk D_j with radius $d(j, X)$ and center in the point where this client is located. Let us consider the resulting intersection of two or more such disks. These disks and their intersections are called *regions.* The total number of regions is big, but we can eliminate some, and consider the convex regions as those defined by intersections only. Thus, we have at most n^2 regions. Now we define a binary matrix (a_{kj}) to indicate the clients which will patronize a facility of the follower if it is opened inside of a region. Formally, define $a_{kj} := 1$ if a facility of the follower in region k

captures the client j and $a_{kj} := 0$ otherwise. In order to present the follower problem as an integer linear program we introduce two sets of the decision variables:

$$y_k = \begin{cases} 1 & \text{if the follower opens his own facility inside of region } k, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_j = \begin{cases} 1 & \text{if the follower captures client } j, \\ 0 & \text{otherwise.} \end{cases}$$

Now the follower problem can be written as the maximum capture problem:

$$\max \sum_{j=1}^{n} w_j z_j$$

$$\text{subject to} \qquad z_j \leq \sum_{k=1}^{n^2} a_{kj} y_k, \quad j = 1, \ldots, n,$$

$$\sum_{k=1}^{n^2} y_k = r,$$

$$y_k, z_j \in \{0, 1\}.$$

The objective function gives the market share of the follower. The first constraint guarantees that client j will patronize a facility of the leader only if the follower has no facility at the distance less than $d(j, X)$. The second constraint allows the follower to open exactly r facilities.

In our computational experiments we observe that the integrality gap is small for this problem in the case of the two-dimensional Euclidean plane. The branch and bound method easily finds an optimal solution. For this reason, the exact value $W^*(X)$ is used in our heuristic for the centroid problem. Note that the follower problem is polynomially solvable for fixed r.

4 Subproblem for a facility of the leader

Let us consider the $(r \mid X_{p-1} + 1)$-centroid subproblem where the leader has a set of $p - 1$ facilities and want to open another one facility in the best position. We claim that there is a small number of points in the Euclidean plane which

we should check for finding this best position. For such points we solve the follower problem and choose the point with maximal leader market share.

For each client j we have disk D_j with radius $R_j = d(j, X)$. If we move the new facility, some regions are modified. Note, that all points inside a region are equivalent for the follower. We will get a new instance of the follower problem if a new region is created or an old region is vanished. Thus, we should consider these cases only. Further, we need the following well-known result for the Euclidean spaces.

Theorem 1. (Helly) *Suppose that $G_1, \ldots, G_k$ is a finite collection of convex subsets of d-dimensional Euclidean space, where $k > d$. If the intersection of every $d+1$ of these sets is nonempty, then the whole collection has a nonempty intersection.*

Theorem 2. *The $(r \mid X_{p-1} + 1)$-centroid problem is polynomially solvable for fixed r.*

Sketch of the proof. By Theorem 1, we can restrict our attention by the triples of the clients and corresponding disks. Let D_1, D_2, D_3 be the disks with radii R_1, R_2, R_3 respectively and the leader opens a new facility. Consider the simplest case: $D_1 \cap D_2 \neq \emptyset$, $D_1 \cap D_3 = \emptyset$, $D_2 \cap D_3 = \emptyset$ (see Fig.1). The region $D_1 \cap D_2$ allows the follower capturing clients 1, 2. To prevent it, new facility of the leader may be opened on the straight line between clients 1 and 2 or in disk D_1' with radius $R_1' = d(1, 2) - R_2$ or in disk D_2' with radius $R_2' = d(1, 2) - R_1$. The case $D_1 \cap D_2 \neq \emptyset$, $D_1 \cap D_3 \neq \emptyset$, $D_2 \cap D_3 \neq \emptyset$ but $D_1 \cap D_2 \cap D_3 = \emptyset$ is similar to the previous one (see Fig.2).

Now we consider the most interesting case: $D_1 \cap D_2 \cap D_3 \neq \emptyset$ (see Fig.3). To destroy this region, the leader can open new facility in disk D_1' or in disk D_2' or in disk D_3' or in the shaded nonconvex region.

Note that all points on the dotter lines and in the shaded region are identical for the new facility. Hence, we may consider these lines and shaded regions and their intersections only. Number the lines and regoins is polynomially bounded and we get the desired. □
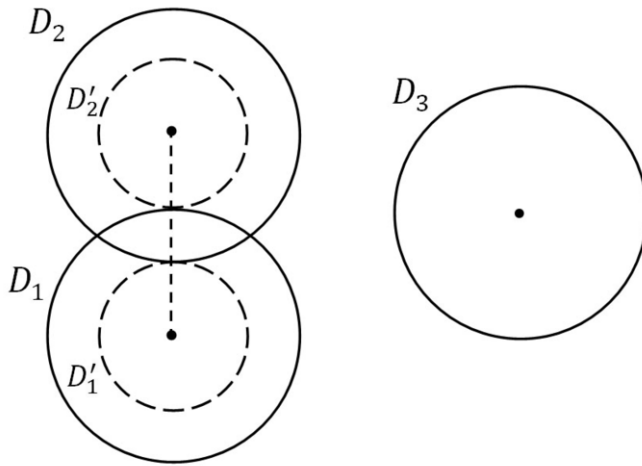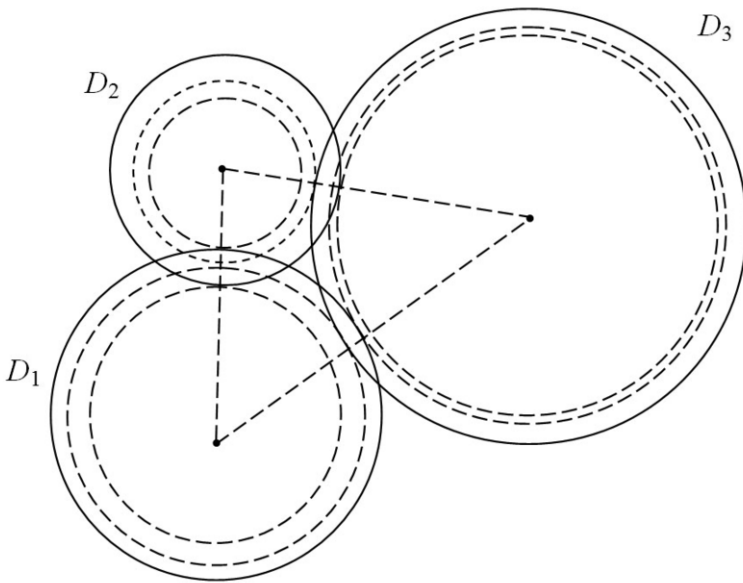
Fig. 1. The simplest case
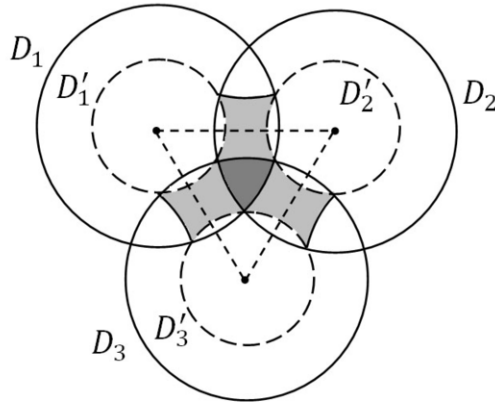

Fig. 2. The case $D_1 \cap D_2 \cap D_3 = \varnothing$

Fig. 3. The case $D_1 \cap D_2 \cap D_3 \neq \varnothing$

5   Computational experiments

We use the obtained results for the local search under the Swap neighborhood. We apply the VNS framework [5] where (k, l)-Swap neighborhoods are used with different values k and l. In this neighborhoods we move k facilities to new positions but not far than the distance l from the current positions. The values $l = 100, 150, 200, 250, 300$ and $k = 1, 2, 3$ are used at the shaking step and $l = 50$, $k = 1$ at the local improvement step of the method.

We have coded the VNS algorithm in Delphi 7.0 environment and tested it on benchmark instances from the electronic library *Discrete Location Problems*. For all instances we have $n = 50$, and demand points are randomly distributed among the square $7000{\times}7000$ uniformly. Two types of weights are considered: $w_j = 1$ and $w_j \in [1, 200]$. For all instances the behavior of the algorithm with $p = r = 10$ is studied. Table 1 shows the computational results for 10 instances. The second column of the Table 1 presents the market share of the leader according to the alternating heuristic from [1]. In brackets these values are shown as percentages. The third column shows the same values for the alternating heuristic with clustering from [2]. The last column presents the leader market share for the VNS algorithm. As we can see, the local search approach based on the discretization result for $(r|X_{p-1} + 1)$-centroid problem is useful and can increase the leader market share. The same conclusions were obtained in the case $w_j = 1$.

Table 1
Market share of the leader, $w_j \in [1, 200]$

Instance number	Alternating heuristic	Procedure of clustering	VNS algorithm
111	1404 (31%)	1671 (37%)	1860 (41%)
211	1591 (28%)	1992 (35%)	2250 (40%)
311	1379 (29%)	1756 (37%)	1948 (41%)
411	1541 (29%)	1917 (36%)	2106 (40%)
511	1418 (31%)	1668 (37%)	1996 (40%)
611	1234 (27%)	1735 (38%)	1874 (42%)
711	1512 (27%)	1918 (34%)	2215 (40%)
811	1318 (26%)	1803 (36%)	1933 (39%)
911	1375 (27%)	1868 (35%)	2222 (42%)
1011	1467 (29%)	1875 (37%)	2061 (41%)

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