A new alternating heuristic for the $(r \mid p)$–centroid problem on the plane

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Abstract In the $(r \mid p)$-centroid problem, two players, called leader and follower, open facilities to service clients. We assume that clients are identified with their location on the Euclidian plane, and facilities can be opened anywhere in the plane. The leader opens p facilities. Later on, the follower opens r facilities. Each client patronizes the closest facility. Our goal is to find p facilities for the leader to maximize his market share. For this Stackelberg game we develop a new alternating heuristic, based on the exact approach for the follower problem. At each iteration of the heuristic, we consider the solution of one player and calculate the best answer for the other player. At the final stage, the clients are clustered, and an exact polynomial-time algorithm for the $(1 \mid 1)$-centroid problem is applied. Computational experiments show that this heuristic dominates the previous alternating heuristic of Bhadury, Eiselt, and Jaramillo.

1 Introduction

This paper addresses a Stackelberg facility location game on a two–dimensional Euclidian plane. It is assumed that the clients demands are concentrated at a finite number of points in the plane. In the first stage of the game, a player, called the leader, opens his own p facilities. At the second stage, another player, called here the follower, opens his own r facilities. At the third final stage, each client chooses the closest opened facility as a supplier. In case of ties, the leader's facility is preferred. Each player tries to maximize his own market share. The goal of the game is to find p points for the leader facilities to maximize his market share.

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This Stackelberg game was studied by Hakimi in 1981 [8, 9] for location on a network. Following Hakimi, the leader problem is called a *centroid problem* and the follower problem is called a *medianoid problem.* In [1] the centroid problem with another behavior of clients was considered. In [7] an exact polynomial time algorithm is presented for these problems in case $p = r = 1$. Similar models with locational constraints are studied in [3,4]. For arbitrary p and r, an alternating heuristic is presented in [2]. A greedy and a minimum-differentiation algorithm are used for approximation of the follower market share. A comprehensive review of complexity results and properties of the problems can be found in [10, 11, 13].

In this paper we improve the alternating heuristic from [2] using an exact approach for the follower problem. We reduce it to the discrete maximum capture problem and apply the branch and bound method. At the end of the alternating process, the final solution for the leader is improved by using an exact algorithm for the $(1 \mid 1)$-centroid problem. All clients are clustered into p subsets. For each subset we relocate the leader facility using the optimal solution for the $(1 \mid 1)$-centroid problem. Computational results for randomly generated instances [6] show that the new approach dominates the benchmark procedures.

2 Mathematical model

Let us consider a two–dimensional Euclidian plane in which n clients are located. We assume that each client j has a positive demand w_j. Let X be the set of p points where the leader opens his own facilities and let Y be the set of r points where the follower opens his own facilities. The distances from client j to the closest facility of the leader and the closest facility of the follower are denoted as $d(j,X)$ and $d(j,Y)$ respectively. The client j prefers Y over X if $d(j,Y) < d(j,X)$ and prefers X over Y otherwise. By

$$U(Y \prec X) := \{ j \mid d(j,Y) < d(j,X)\}$$

we denote the set of clients preferring Y over X. The total demand captured by the follower by locating his facilities at Y while the leader locates his facilities at X is given by

$$W(Y \prec X) := \sum (w_j \mid j \in U(Y \prec X)).$$

For X given, the follower tries to maximize his own market share. The maximal value $W^*(X)$ is defined to be

$$W^*(X) := \max_{Y, |Y|=r} W(Y \prec X).$$

This maximization problem will be called the *follower problem.* The leader tries to minimize the market share of the follower. This minimal value $W^*(X^*)$ is defined to be

$$W^*(X^*) := \min_{X, |X|=p} W^*(X).$$

For the best solution X^* of the leader, his market share is $\sum_{j=1}^{n} w_j - W^*(X^*)$. In the $(r \mid p)$–*centroid problem,* the goal is to find X^* and $W^*(X^*)$.

3 The follower problem

Let us first describe an exact approach for the follower problem. Such problem will be rewritten as an integer linear programming problem, and solved using a branch and bound method.

For each client j, we introduce a disk D_j with radius $d(j,X)$ and center in the point where this client is located. Let us consider the resulting intersection of each set of two or more such disks. These disks and their intersections will be called *regions*. The total number of regions is large, but we can eliminate some, and consider the maximal regions as those defined by intersections only. In any case, we have at most n^2+n regions. Now we define a binary matrix (a_{kj}) to indicate the clients which will patronize a facility of the follower if it is opened inside a region. Formally, define $a_{kj} := 1$ if a facility of the follower in region k captures the client j and $a_{kj} := 0$ otherwise. In order to present the follower problem as an integer linear program we introduce two sets of the decision variables:

$$y_k = \begin{cases} 1 & \text{if the follower opens his own facility inside of region } k, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_j = \begin{cases} 1 & \text{if the follower captures client } j, \\ 0 & \text{otherwise.} \end{cases}$$

Now the follower problem can be written as the maximum capture problem:

$$\max \sum_{j=1}^{n} w_j z_j$$

$$\text{subject to} \qquad z_j \le \sum_{k=1}^{n^2+n} a_{kj} y_k, \quad j = 1,\dots,n,$$

$$\sum_{k=1}^{n^2+n} y_k = r,$$

$$y_k, z_j \in \{0,1\}.$$

The objective function gives the market share of the follower, to be minimized. The first constraint guarantees that client j will patronize a facility of the leader only if the follower has no facility at the distance less than $d(j,X)$. The second constraint allows the follower to open exactly r facilities.

In [2] it is claimed that the problem is NP-hard, and two heuristics are developed. We note that the integrality gap is small for this problem in the case of the two-dimensional Euclidian plane. The branch and bound method [5] easily finds the optimal solution. For this reason, the exact value $W^*(X)$ is used in our heuristic for the centroid problem.

4 Alternating heuristic

In this section we present an improved alternating heuristic. The idea of alternating methods is well-known [2, 12]. Given a solution X for the leader, the best-possible solution Y for the follower is computed. Once that is done, the leader may tentatively assume the role of the follower and reoptimize his set of facilities by solving the corresponding problem for the given solution Y. This process is then repeated until a termination condition is satisfied. In other words, the players alternately solve a follower problem. Convergence results of similar alternating algorithms for equilibrium problems can be found in [14].

In our case, a key issue is that an exact-polynomial time method by Drezner [7] for the $(1 \mid 1)$-centroid problem is applied. The method is described as follows.

Improved alternating heuristic

1. Create a starting solution X for the leader.
2. **While** not termination condition **do**:
    - 2.1. Find the best solution Y for the follower against the solution X.
    - 2.2. Find the best solution X for the leader against the solution Y.

   end while
3. Improve the final solution X by solving exactly the $(1 \mid 1)$–centroid problem.

The starting solution is generated at random. Calculations are terminated after a sufficiently large number of iterations. Note that the optimal solution of the follower problem shows us a subset of regions only. However we need the exact coordinates for the facilities. For the follower, all points inside each region are equivalent. But it is not the case for the leader at Step 2.2. of alternating process.

In order to minimize the market share of the follower, we should compute the coordinates of the leader facilities inside of the regions very carefully. To reduce the running time of the iterative process, we take the center points of the regions. At the final iteration the current solution X is modified as follows. All clients are clustered in p subsets according to X. Clients are allocated to the same subset if their closest leader facility is the same. For each subset, the $(1 \mid 1)$–centroid problem is solved, assuming that the follower will attack each subset by opening one facility. Optimal solutions for these subsets generate a new solution for the leader. As a result, we may get a new clustering of the clients and the procedure is repeated. The best found solution is the result of the method.

5 Computational experiments

We have coded the improved alternating algorithm in Delphi 7.0 environment and tested it on benchmark instances from the electronic library *Discrete Location Problems* [6]. For all instances we have $n = 100$, and demand points are randomly distributed among the square 7000×7000 uniformly. Two types of weights are considered: $w_j = 1$ and $w_j \in [1, 200]$. For all instances the behavior of the algorithm with $p = r$ is studied.

Two types of experiments were performed. In the first experiment we wanted to measure the influence of the starting solution of the leader at Step 1 of the algorithm. Different random solutions and *corner* solutions when all facilities are concentrated near a corner of the square were created. For all cases we observe that the algorithm produces the same final facilities locations for the leader for both types of weights. We guess that for higher dimensions, $n > 100$ and $p \neq r$, we may get another behavior of the algorithm, but now we are observing a fast convergence to the same equilibrium.

Table 1 Comparison of alternating heuristics

Instance number	Heuristic of Eiselt et al.	Improved heuristic	Procedure of clustering
111	1404 (31%)	1581 (35%)	1671 (37%)
211	1591 (28%)	1820 (32%)	1992 (35%)
311	1379 (29%)	1662 (35%)	1756 (37%)
411	1541 (29%)	1749 (33%)	1917 (36%)
511	1418 (31%)	1574 (35%)	1668 (37%)

In the second experiment our algorithm and the alternating heuristic from [2] are compared. Our goal is to understand the influence of the exact approach for the follower problem at Step 2 and idea of clustering at Step 3. Table 1 presents the computational results for the case $w_j \in [1, 200]$, $p = r = 10$. The second column of the Table 1 presents the market share of the leader according to [2]. Actually, we apply this algorithm to create a solution for the leader and then the exact value of the leader market share is computed. We show in brackets such values as percentages. The third column shows the same values for our algorithm without Step 3. The last column presents the leader market share for the final solution. As we can see, the exact method for the follower problem and the clustering procedure are important and they increase the leader market share. The same conclusions were obtained in the case $w_j = 1$. These values seem to be optimal, though we do not have a proof. Constructing sharp upper bounds for the global optimum is a very interesting and important direction for further research.

6 Conclusions

We have considered the well-known Stackelberg facility location game on the two-dimensional Euclidian plane. An improved alternating heuristic is presented. In this new heuristic, we have used the exact method for the follower problem, and a clustering procedure with an exact polynomial-time method for the $(1 \mid 1)$-centroid problem is used. Computational results for random generated instances show advantages of the proposed approach against benchmark procedures.

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References

1. Bauer, A., Domshke, W., Pesch, E.: Competitive location on a network. European Journal of Operational Research. **66**, 372-391 (1993)
2. Bhadury, J., Eiselt, H.A., Jaramillo, J.H.: An alternating heuristic for medianoid and centroid problems in the plane. Computers and Operations Research. **30**, 553-565 (2003)
3. Carrizosa, E., Conde, E., Munoz-Marquez, M., Puerto, J.: Simpson points in planar problems with locational constraints. The round-norm case. Mathematics of Operations Research. **22**, 276–290 (1997)
4. Carrizosa, E., Conde, E., Munoz-Marquez, M., Puerto, J.: Simpson points in planar problems with locational constraints. The polyhedral–gauge case. Mathematics of Operations Research. **22**, 291–300 (1997)
5. CPLEX: http://www-142.ibm.com/software/products/ru/ru/ilogcple/
6. *Discrete Location Problems*. Benchmark library. http:// math.nsc.ru/AP/benchmarks/index.html
7. Drezner, Z.: Competitive location strategies for two facilities. Regional Science and Urban Economics. **12**, 485–493 (1982)
8. Hakimi S.L.: On locating new facilities in a competitive environment. ISOLDE Conference, Skodsborg, Denmark, (1981)
9. Hakimi S.L.: On locating new facilities in a competitive environment. European Journal of Operational Research. **12**, 29–35 (1983)
10. Hakimi, S.L.: Locations with spatial interactions: competitive locations and games, in: P. Mirchandani, R. Francis (Eds.) *Discrete Location Theory*, Wiley, 439–478 (1990)
11. Hansen, P., Thisse, J.F., Wendell, R.E.: Equilibrium analysis for voting and competitive location problems, in: P. Mirchandani, R. Francis (Eds.) *Discrete Location Theory*, Wiley, 479–502 (1990)
12. Hotelling, H.: Stability in competition. Economic J. **39**, 41–57 (1929)
13. Kress, D., Pesch, E.: Sequential competitive location on networks. European Journal of Operational Research. (in print)
14. Sherali, H.D., Soyster, A.L.: Convergence analysis and algorithmic implications of twodynamic processes toward an oligopoly — competitive fringe equilibrium set. Computers and Operations Research. **15**, 69-81 (1988)