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INVITED PAPER

On the complexity of the $(r\,|\,p)$-centroid problem in the plane

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Abstract In the $(r \mid p)$-centroid problem, two players, called leader and follower, open facilities to service clients. We assume that clients are identified with their location on the Euclidean plane, and facilities can be opened anywhere in the plane. The leader opens p facilities. Later on, the follower opens r facilities. Each client patronizes the closest facility. In case of ties, the leader's facility is preferred. The goal is to find p facilities for the leader to maximize his market share. We show that this Stackelberg game is Σ_2^P-hard. Moreover, we strengthen the previous results for the discrete case and networks. We show that the game is Σ_2^P-hard even for planar graphs for which the weights of the edges are Euclidean distances between vertices.

Keywords Competitive location · Bilevel programming · Leader–Follower problem

Mathematics Subject Classification 90B80 · 90C05 · 90C27

1 Introduction

This paper addresses a Stackelberg facility location game in a two-dimensional Euclidean plane. It is assumed that the clients demands are concentrated at a finite number of points in the plane. In the first stage of the game, a player, called the leader, opens his p facilities. At the second stage, another player, called here the follower, opens own r facilities. At the third final stage, each client chooses the closest opened facility as a supplier. In case of ties, the leader's facility is preferred. Each player tries to maximize his market share. The goal of the game is to find p points for the leader facilities to maximize his market share.

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This Stackelberg game was studied by Hakimi in 1981 (Hakimi 1981, 1990) for location on a network. Following Hakimi, the leader problem is called a *centroid problem* and the follower problem is called a *medianoid problem.* A comprehensive review of complexity results and properties of the problems can be found in Kress and Pesch (2012).

Computational complexity of the centroid problem on general graphs is studied in Noltemeier et al. (2007), Spoerhase (2012). It is shown that the problem is Σ_2^P-hard for two cases:

1. the discrete case when facilities can be opened in vertices of the graph,
2. the absolute case when facilities can be opened in vertices and anywhere on edges.

In this paper, we show that the problem is Σ_2^P-hard for the two-dimensional Euclidean plane. Moreover, we improve the previous results for the discrete and absolute cases. We show that the problem is Σ_2^P-hard even for planar graphs for which the weights of edges are Euclidean distances between the corresponding points in the plane. The follower problem for these cases is NP-hard in the strong sense. Some applications of these models can be found in Beresnev and Suslov (2010), Kress and Pesch (2012).

2 Mathematical model

Let us consider a two-dimensional Euclidean plane in which n clients are located. We assume that each client j has a positive demand w_j. Let X be the set of p points where the leader opens his facilities and let Y be the set of r points where the follower opens his facilities. The distances from client j to the closest facility of the leader and the closest facility of the follower are denoted as $d(j, X)$ and $d(j, Y)$, respectively. The client j prefers Y over X if $d(j, Y) < d(j, X)$ and prefers X over Y otherwise. The set of clients preferring Y over X is denoted by $U(Y \prec X) := \{j \mid d(j, Y) < d(j, X)\}$. The total demand captured by the follower by locating his facilities at Y while the leader locates his facilities at X is given by $W(Y \prec X) := \sum(w_j \mid j \in U(Y \prec X))$.

For X given, the follower tries to maximize his market share. The maximum value $W^*(X)$ is defined to be

$$W^*(X) := \max_{Y,|Y|=r} W(Y \prec X).$$

This maximization problem will be called the *follower problem.* The leader tries to minimize the market share of the follower. This minimum value $W^*(X^*)$ is defined to be

$$W^*(X^*) := \min_{X,|X|=p} W^*(X).$$

For the best solution X^* of the leader, his market share is $\sum_{j=1}^n w_j - W^*(X^*)$. In the $(r \mid p)$-*centroid problem*, the goal is to find X^* and $W^*(X^*)$.

3 Computational complexity

Let us consider the following decision problem. We are given two vectors $x = (x_1, \dots, x_l)$ and $y = (y_1, \dots, y_k)$ of Boolean variables and a formula $\varphi(x, y)$ in the disjunctive normal form. We need to check whether the formula $\exists x_1, \dots, \exists x_l$ $\forall y_1, \dots, \forall y_k\ \varphi(x, y)$ is satisfied. It is known that this problem is Σ_2^P-complete even each term of φ consists of exactly three literals (Schaefer and Umans 2002). We will reduce this decision problem to the $(r|p)$-centroid problem in the plane. To this end, we consider a special case of φ where each term contains exactly one x-variable and two or three y-variables. This decision problem is denoted by $\exists\forall 3, 4Sat$.

Lemma 1 *The problem $\exists\forall 3, 4Sat$ is Σ_2^P-complete.*

Proof Let us consider a term $(x_1 \wedge x_2 \wedge y_1)$. We can replace it with formula $\forall y'(x_1 \wedge y' \wedge y_1) \vee (x_2 \wedge \neg y' \wedge y_1)$ using a new variable y'. It is easy to see that this formula is *True* if and only if the term is *True*.

Assume that φ contains a term $(y_1 \wedge y_2 \wedge y_3)$. In this case, we introduce a new variable x' and replace the term with a formula $\exists x'(x' \wedge y_1 \wedge y_2 \wedge y_3)$. □

Theorem 1 *The $(r \mid p)$-centroid problem in the plane is Σ_2^P-hard.*

Proof In the reduction of the decision problem $\exists\forall 3, 4Sat$ to the $(r \mid p)$-centroid problem each variable x_i or y_j will be represented by a circuit of circles (see Fig. 1). A radius of each circle is 1. Two clients are associated with each circle. One of them is located in the center of circle. He has a positive weight w_x or w_y depending on the type of variable. The second client is located on the boundary of the circle. He has weight W. We assume that $W > 2w_y > w_x > w_y$. The distance between centers of the neighboring circles is equal to $2 - \varepsilon$ for some small positive ε. The number of circles in the circuit is even, say $2q$. If the circuit corresponds to y_j and $p = 2q$, $r = q$, then the optimal solution for the leader is to open facilities on the boundary of all circles where clients with weight W are located. In this case, his market share is pW. The optimal value of the follower is pw_y. It can be obtained by two different ways. If the follower opens a facility in the intersection of two circles, he captures two clients at the centers. There are two different partitions of the circuit into the pairs of neighboring circles. One of them is to correspond to the *True* value of y_j, another one—*False* value. A similar idea is used to prove the complexity results for the Euclidean p-center problem (Megiddo and Supowit 1994).

Let us consider an x_i circuit and assume that $p = 3q$. Then the optimal solution for the leader is to open $2q$ facilities on the boundary of all circles where clients with weight W are located and open q facilities in the middle points of every second intersection of circles. Thus, it will make impossible for the follower to capture more than one client with weight w_x by one facility. Since $2w_y > w_x$, the follower will try to capture the clients for the y circuits instead of the x circuit. As we will see later, all clients in x circuits and all clients with weight W in y circuits will be captured by the leader. All clients in the centers of circles in y circuits will be captured by the follower.

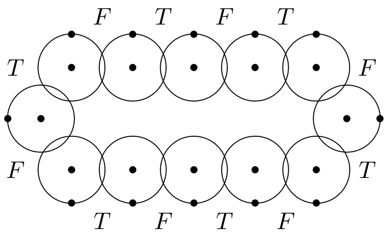

Fig. 1 A circuit for a Boolean variable

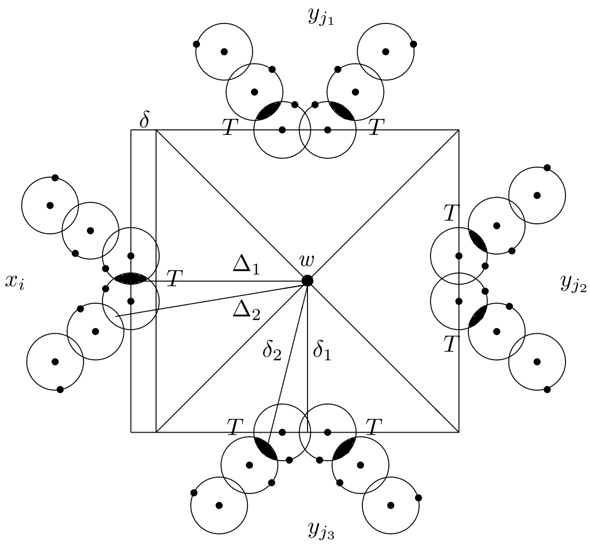

Fig. 2 A term configuration of circuits

For each term $(x_i \wedge y_{j_1} \wedge y_{j_2} \wedge y_{j_3})$, we introduce an additional client with a positive weight w, $w < w_y$. This client is located at the same distance from $y_{j_1}, y_{j_2}, y_{j_3}$ circuits and slightly farther, by a positive δ, from x_i circuit (see Fig. 2). Let us denote the distance from this client to the central point of intersection of nearest circles for x_i circuit by Δ_1. A similar distance from this client to the central point of intersection of nearest circles for $y_{j_1}, y_{j_2}, y_{j_3}$ circuits we denote by δ_1. Now we consider the intersection of one nearest circle with another neighboring circle. In Fig. 2, these intersections for $y_{j_1}, y_{j_2}, y_{j_3}$ circuits are marked as black regions. The distance from the client to the central point of the intersection for x_i circuit is denoted by Δ_2 and a similar distance for $y_{j_1}, y_{j_2}, y_{j_3}$ circuits is denoted by δ_2. We assume that $\delta_1 < \Delta_1 < \delta_2 < \Delta_2$. This configuration is possible in the Euclidean plane for large δ_1.

If the *True* assignment for the variables $x_i, y_{j_1}, y_{j_2}, y_{j_3}$ corresponds to Fig. 2 and the players open facilities in the black regions, then the client patronizes the leader facility, since $\Delta_1 < \delta_2$. For other assignments for the variables, the client patronizes the follower facility, since $\delta_1 < \Delta_1$ and $\delta_2 < \Delta_2$.

In other words, exactly one assignment (indicated in Fig. 2) corresponds to the case when the client patronizes the leader facility. In this case, the term will be *True*. If the leader will capture at least one such a client, the formula is satisfied. A schematic

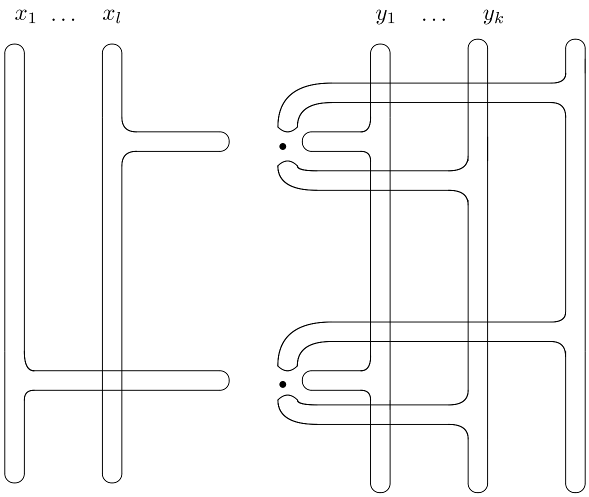

Fig. 3 A schematic plan of the reduction

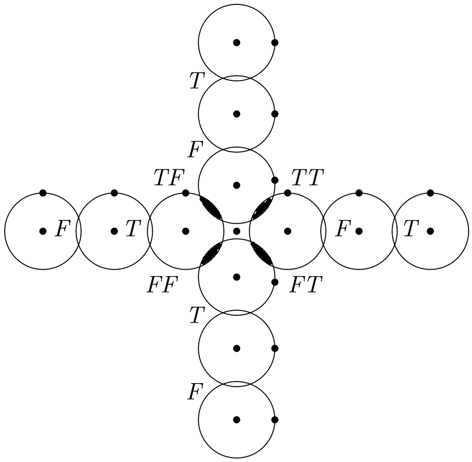

Fig. 4 A junction

plan of the allocation of these clients and their relationship with the circuits is shown in Fig. 3.

As we can see, the circuits have mutual intersections or junctions. But the junctions correspond to pairs $x_i x_j$ or $y_i y_j$ only. We have no junction for pair $x_i y_j$. Let us consider a junction in details and present a configuration to save the parity. To this end, we introduce an additional client with weight w_x or w_y and put it in the center of the junction (see Fig. 4).

We claim that the optimal solution for the instance of the $(r \mid p)$-centroid problem indicates whether the formula $\exists\forall 3, 4Sat$ is satisfied or not. Let m be the number of terms, p_x be the number of circles for x circuits, q_x be the number of their junctions, p_y and q_y be the number of circles and junctions for y. We put $r = 0.5p_y$, $p = p_y + 1.5p_x$. In this case, the leader will use $p_x + p_y$ facilities to capture all clients with weight W. By the other $0.5p_x$ facilities, the leader will capture p_x clients in the center of circles for x circuits and q_x clients for junctions. The rest $m + p_y + q_y$ clients will be distributed between the leader and the follower. But $p_y + q_y$ clients will be captured by the follower, because $w < w_y$. The m clients for the terms will patronize the leader or the follower facilities depends on the parity of the solution. It is easy to see that the formula is satisfied if and only if the leader will get at least one of them.

Now we show that the reduction is polynomial in l, k, m. In Fig. 3, we can see that the plan is decomposed on m separated regions, one for each term. In each region, we have the circles for y circuits on the right side. The number of these circles is linear in k. Similarly, the number of circles for x is linear in l. Without loss of generality, we can assume that centers of all circles have coordinates with a polynomial encoding length. Otherwise, we can slightly move the centers of circles by varying ε, which completes the proof. □

Corollary 1 *The discrete $(r \mid p)$-centroid problem is Σ_2^P-hard even the clients and facilities are placed in the two-dimensional Euclidean plane.*

The proof of the statement straight follows from the previous reduction if we define the set of possible facility locations as a union of the set of clients with weight W and the set of central point for intersections of the circles.

Corollary 2 *The $(r \mid p)$-centroid problem on a network is Σ_2^P-hard even for planar graphs with vertices in the two-dimensional Euclidean plane and weights of the edges are Euclidean distances between corresponding points.*

Proof We slightly modify the previous reduction and introduce a network in the following way. Each client from previous construction generates a vertex of the network. Two vertices are adjacent if

- they are centers of the circles and the circles have nonempty intersection or
- one vertex is the center of circle, another vertex is located on the boundary of the same circle.

For each term, we introduce four or three dummy vertices. The number of the vertices depends on the number of literals in the term. We put the dummy vertices into intersections of the nearest circles to the client with weight w (see Fig. 5). Each dummy vertex is adjacent with the client and with centers of these circles.

For each junction, we also introduce four dummy vertices, one vertex for each region *TF*, *FT*, *TT*, *FF*. Again, each dummy vertex is adjacent to central vertex and with centers of nearest circles (see Fig. 6). The weight of each dummy vertex is 0. It is easy to check that our graph is planar. Each optimal solution of the centroid problem for the graph is optimal one for the plane. □

We can use the same idea to study computational complexity of the follower problem. To this end, we reduce the well-known 3Sat problem to the follower problem. In the 3Sat problem, we have a Boolean formula in the conjunctive normal form. Each clause includes exactly three literals. We need to decide whether this formula is satisfied.

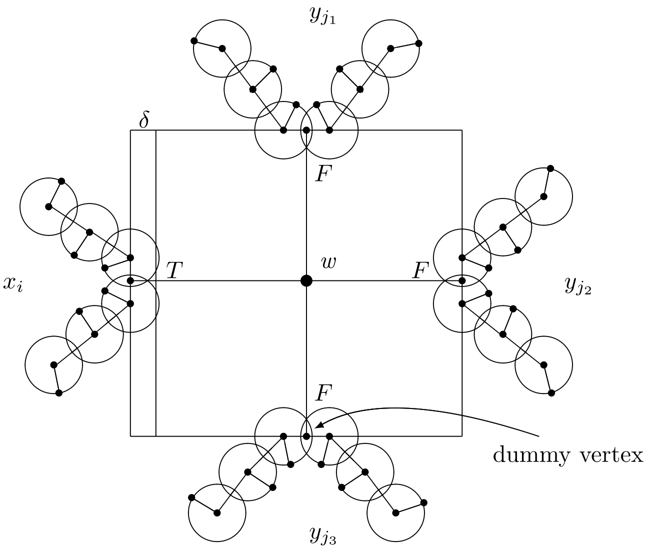

Fig. 5 A fragment of the network for a term

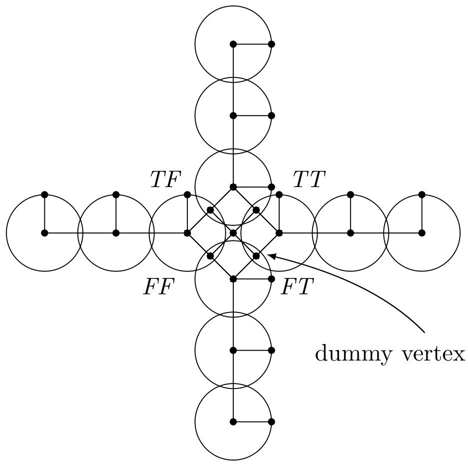

Fig. 6 A fragment of the network for a junction

Theorem 2 *The follower problem in the two-dimensional Euclidean plane is NP-hard in the strong sense.*

Proof We modify the previous reduction for Fig. 2 and Fig. 3 only. Again, for each Boolean variable y_j, we create a circuit which consists of the circles with radius 1. But for each clause, we create two clients. One of them has weight W, another one has weight w (see Fig. 7). The distance between these two clients is greater than the distance from the client with weight w to the intersection of nearest circles for corresponding circuits.

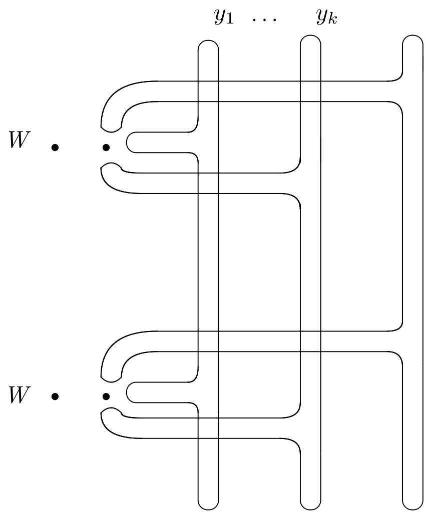

Fig. 7 A modified plan of the reduction

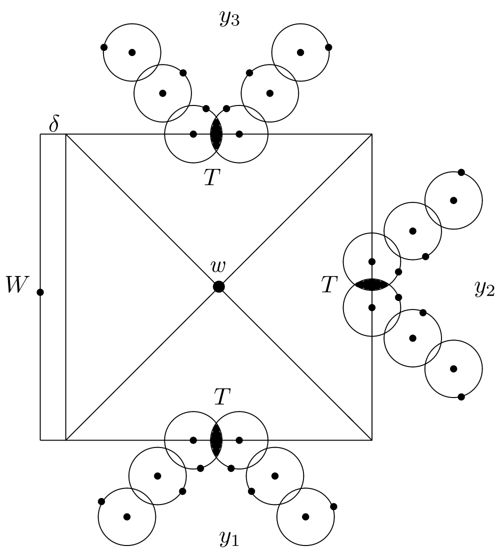

Fig. 8 A clause configuration in the reduction for the follower problem

Figure 8 shows a configuration for clause $(y_1 \vee y_2 \vee y_3)$. If at least one variable is *True*, then the client located in the center of square patronizes the follower facility in a black region. Otherwise, the client will be captured by the leader.

Put $p = p_y + m$, $r = 0.5p_y$. The leader opens facilities where clients with weight W are located. The leader's market share is at least $W(p_y + m)$. The follower opens facilities in the intersections of circles and get at least $(p_y + q_y)w_y$. The rest m clients with weight w are distributed between the leader and the follower. If the follower will capture all these clients, the formula is satisfied. Hence, the follower problem is NP-hard. Now we wish to show that it is NP-hard in the strong sense.

In the follower problem, we need exact coordinates (z_1, z_2) for each client. Let us return to the schematic plan of the reduction (see Fig. 7). It is easy to see that there are two constants c_1 and c_2, such that $z_1 \le c_1 k$ and $z_2 \le c_2 m$ for all clients. Hence, if we put $W = 3$, $w_y = 2$, $w = 1$, then we get the desired. □

Corollary 3 *The discrete follower problem is NP-hard in the strong sense even if the clients and facilities are placed in the two-dimensional Euclidean plane.*

Corollary 4 *The follower problem on a network is NP-hard in the strong sense even for planar graphs with vertices in the two-dimensional Euclidean plane and weights of the edges are Euclidean distances between corresponding points.*

4 Conclusions

In this paper, we have considered the $(r \mid p)$-centroid problem in the plane. It is shown that the problem is Σ_2^P-hard and the follower problem is NP-hard in the strong sense. Moreover, we have strengthened the previous results from Noltemeier et al. (2007), Spoerhase (2012). It is shown that the discrete and absolute $(r \mid p)$-centroid problems are Σ_2^P-hard even for planar graphs with Euclidean weights of the edges.

For further research, it is interesting to study exact and heuristic methods for these extremely difficult problems. The first steps in this direction are made in Alekseeva et al. (2010), Carrizosa et al. (2011), Bhadury et al. (2003), Rodriguez and Perez (2008), Roboredo and Pessoa (2013). For the discrete $(r \mid p)$-centroid problem, the optimal solution can be found for $n \le 100$, $p = r \le 15$ (Alekseeva and Kochetov 2013; Roboredo and Pessoa 2013). Exact methods for the problem in the plane or networks are unknown. Only a few specific cases are tractable (Drezner 1982; Spoerhase 2012). We guess that local search methods, metaheuristics, and matheuristics can be useful in this direction.

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References

Alekseeva E, Kochetova N, Kochetov Y, Plyasunov A (2010) Heuristic and exact methods for the discrete $(r|p)$-centroid problem. Lect Notes Comput Sci 6022:11–22

Alekseeva E, Kochetov Y (2013) Matheuristics and exact methods for the discrete $(r|p)$-centroid problem. In: Talbi E, Brotcorne L (eds) Metaheuristics for bi-level optimization. Springer, Berlin (in press)

Beresnev VL, Suslov VI (2010) A mathematical model of market competition. J Appl Ind Math 4(2):147–157

Carrizosa E, Davydov I, Kochetov Y (2011) A new alternating heuristic for the $(r|p)$-centroid problem on the plane. In: Klatte D, Lüthi H-J, Schmedders K (eds) Operations research proceedings 2011. Springer, Berlin, pp 275–280

Drezner Z (1982) Competitive location strategies for two facilities. Reg Sci Urban Econ 12:485–493

Bhadury J, Eiselt HA, Jaramillo JH (2003) An alternating heuristic for medianoid and centroid problems in the plane. Comput Oper Res 30:553–565

Hakimi SL (1981) On locating new facilities in a competitive environment. In: ISOLDE conference, Skodsborg, Denmark

Hakimi SL (1990) Locations with spatial interactions: competitive locations and games. In: Mirchandani PB, Francis RL (eds) Discrete location theory. Wiley, New York, pp 439–478

Kress D, Pesch E (2012) Sequential competitive location on networks. Eur J Oper Res 217:483–499

Megiddo N, Supowit K (1994) On the complexity of some common geometric location problems. SIAM J Comput 13(1):182–196

Noltemeier H, Spoerhase J, Wirth HC (2007) Muliple voting location and single voting location on trees. Eur J Oper Res 181:654–667

Rodriguez CMC, Perez JAM (2008) Multiple voting location problems. Eur J Oper Res 191(2):437–453

Roboredo MC, Pessoa AA (2013) A branch-and-cut algorithm for the $(r|p)$-centroid problem. Eur J Oper Res 224(1):101–109

Schaefer M, Umans C (2002) Completeness in the polynomial-time hierarchy: Part I: A compendium. ACM SIGACT News 37(33):32–49

Spoerhase J (2012) Competitive and voting location. Ph.D. Thesis, University of Würzburg