The Existence of Equilibria in the Leader-Follower Hub Location and Pricing Problem

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Abstract We propose a model where two competitors, a Leader and a Follower, are sequentially creating their hub and spoke networks and setting prices. The existence of the unique Stackelberg and Nash pricing equilibria is shown. On the basis of these results we give the conclusion about existence of the profit maximising solution for the Leader.

1 Introduction

The Hub Location Problem consists of finding the optimal locations for one or more hubs with respect to some given objective. Because markets are usually oligopolies, the profit of a company is not only affected by the decision of its management, but also by the moves and responses of the competitors. Competition between firms that use hub and spoke networks has been studied mainly from a sequential location approach. An existing firm, the Leader, serves the demand in some region, and a new firm, the Follower, wants to enter. One thing that can strongly affect the competition is the price. In the Facility Location Theory, the pricing has been studied for some time now (some more recent works are [1] and [2]), but that is not the case with

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the hub location problems. Recently, Lüer-Villagra and Marianov in [3] analysed a competitive case of hub location problem where the pricing is taken into account. They argued that a location, or route opening decisions, or even the entrance into a market can be very dependent on the revenues that a company can obtain by operating these locations and routes. In turn, revenues depend on the pricing structure and competitive context.

Here, we consider a sequential hub location and pricing problem in which two competitors, a Leader and a Follower, compete to attract clients in a given market. Each player tends to maximize his own profit rather than a market share. Customers choose which company and route to patronize by price. It is expected that the demand is split according to the logit model. The location of hubs, allocation of spokes, and pricing are to be determined so as to maximize the profit of the Leader. For this Stackelberg competition we show that there are Stackelberg and Nash pricing equilibria, if the networks of the competitors are already set. Besides their existence and uniqueness, transcendental equations for finding both pricing equilibria are provided. On the basis of these results we give the conclusion about existence of the profit maximising solution for the Leader.

2 A Leader-Follower Hub Location and Pricing Problem

The problem is defined over a directed multi-graph $G = G(N,A)$, where N is the non-empty set of nodes and A is the set of arcs. For every arc $(i, j) \in A$, there is an opposite arc $(j,i) \in A$. If a competitor wants to locate a hub at node $i \in N$, that would cost him some fixed amount f_i. Also, hubs can be shared and there are no capacity constraints. For every arc $(i, j) \in A$ there is a fixed (positive) cost g_{ij} for allocating it as a spoke, and a (positive) transport cost per unit of flow c_{ij}. The cost itself is a non-decreasing function of distance. On the inter-hub transfer there is a known fixed discount factor $\alpha \in (0,1)$. At most two hubs are allowed to be on a single route. The transportation cost $c_{ij/kl}$ over a route $i \to k \to l \to j$ is given by the following expression $c_{ij/kl} = c_{ik} + \alpha c_{kl} + c_{lj}$. Demand w_{ij} for every OD pair $(i, j) \in A$ is assumed to be non-elastic and positive. Every customer is served either by the Leader or by the Follower. The logit model is used as a discrete choice model. There are no budget constraints. Following the work [4] and [5], we address the setting where both players are forced to serve all nodes. Now, we introduce the decision variables for the players:

- $x_k = 1$ if the Leader locates a hub at node $k \in N$ and 0 otherwise
- $\lambda_{ij} = 1$ if the Leader establishes a direct connection between nodes $i, j \in N$, where $(i, j) \in A$, and 0 otherwise
- $p_{ij/kl}$ is the price charged by the Leader for the flows between nodes $i \in N$ and $j \in N$, using the intermediate hubs $k,l \in N$.
- $y_k = 1$ if the Follower locates a hub at node $k \in N$ and 0 otherwise
- $\zeta_{ij} = 1$ if the Follower establishes a direct connection between nodes $i, j \in N$, where $(i, j) \in A$, and 0 otherwise

- $q_{ij/kl}$ is the price charged by the Follower for the flows between nodes $i \in N$ and $j \in N$, using the intermediate hubs $k,l \in N$.

The Leader wishes to maximize his profit, anticipating that the Follower will react to his decision by creating own hub and spoke network and own pricing structure. This Stackelberg game can be presented as the following nonlinear mix-integer bilevel optimization problem. The model for the Leader is

$$\max \sum_{i,j,k,l\in N} (p_{ij/kl} - c_{ij/kl}) w_{ij} u_{ij/kl} - \sum_{i\in N} f_i x_i - \sum_{(i,j)\in A} g_{ij}\lambda_{ij} \tag{1}$$

$$\sum_{s,t\in N} x_s x_t \lambda_{is}\lambda_{st}\lambda_{tj} \geq 1, \qquad \forall i,j \in N \tag{2}$$

$$u_{ij/kl} = \frac{x_k x_l \lambda_{ik}\lambda_{kl}\lambda_{lj} e^{-\Theta p_{ij/kl}}}{\sum_{s,t\in N} x_s x_t \lambda_{is}\lambda_{st}\lambda_{tj} e^{-\Theta p_{ij/st}} + \gamma^*_{ij}}, \quad \forall i,j,k,l \in N \tag{3}$$

$$\gamma^*_{ij} = \sum_{s,t\in N} y^*_s y^*_t \zeta^*_{is}\zeta^*_{st}\zeta^*_{tj} e^{-\Theta q^*_{ij/st}}, \qquad \forall i,j \in N \tag{4}$$

$$((y^*_i),(\zeta^*_{ij}),(v^*_{ij/kl}),(q^*_{ij/st})) \in F^*((x_i),(\lambda_{ij}),(u_{ij/kl}),(p_{ij/kl})) \tag{5}$$

$$p_{ij/kl} \geq 0, \qquad \forall i,j,k,l \in N \tag{6}$$

$$x_i \in \{0,1\}, \qquad \forall i \in N \tag{7}$$

$$\lambda_{ij} \in \{0,1\}, \qquad \forall (i,j) \in A \tag{8}$$

Feasible solutions are tuples $((x_i),(\lambda_{ij}),(p_{ij/kl}),(y^*_i),(\zeta^*_{ij}),(q^*_{ij/st}))$ satisfying constraints (1)-(8), where (5) indicates that the Follower's problem has the optimal solution $F^*((x_i),\,(\lambda_{ij}),(u_{ij/kl}),\,(p_{ij/kl}))$ for a particular Leader's solution $((x_i),(\lambda_{ij}),$ $(u_{ij/kl}),(p_{ij/kl}))$. The model for the Follower is

$$\max \sum_{i,j,k,l\,\in\,N} (q_{ij/kl} - c_{ij/kl}) w_{ij} v_{ij/kl} - \sum_{i\,\in\,N} f_i y_i - \sum_{(i,j)\,\in\,A} g_{ij}\zeta_{ij} \tag{9}$$

$$\sum_{s,t\in N} y_s y_t \zeta_{is}\zeta_{st}\zeta_{tj} \geq 1, \qquad \forall i,j \in N \tag{10}$$

$$v_{ij/kl} = \frac{y_k y_l \zeta_{ik}\zeta_{kl}\zeta_{lj} e^{-\Theta q_{ij/kl}}}{\sum_{s,t\in N} y_s y_t \zeta_{is}\zeta_{st}\zeta_{tj} e^{-\Theta q_{ij/st}} + \eta_{ij}}, \quad \forall i,j,k,l \in N \tag{11}$$

$$\eta_{ij} = \sum_{s,t\in N} x_s x_t \lambda_{is}\lambda_{st}\lambda_{tj} e^{-\Theta p_{ij/st}} \qquad \forall i,j \in N \tag{12}$$

$$q_{ij/kl} \geq 0, \qquad \forall i,j,k,l \in N \tag{13}$$

$$y_i \in \ \{0,1\}, \quad \forall i \in N \tag{14}$$

$$\zeta_{ij} \in \ \{0,1\}, \quad \forall (i,j) \in A \tag{15}$$

The objective functions (1) and (9) are representing the profits of the competitors. Constraints (2) and (10) are assuring that all OD pairs are going to be served. Equations (3) and (11) are representing the Leader's and the Follower's market shares, respectively. Next, (4) is characterizing the effect of the Follower's optimal solution on the Leader's market share. Equation (12) characterizes the Leader's effect on the Follower's market share. In addition, we are distinguishing two extreme cases for the Follower's behaviour: altruistic and selfish.

3 Stackelberg Pricing Equilibrium

We have the optimal pricing expression for the Follower, provided in [3]. Let H_{ij}^L denotes the Leader's set of inter-hub arcs which are connecting OD pair (i,j). Following that, let H_{ij}^{F*} represents the Follower's set of inter-hub arcs that are connecting OD pair (i,j), based on his optimal solution (a hub and spoke topology).

Theorem 1. *(Lüer-Villagra and Marianov (2013)). The Follower's optimal price for every route* $i \to k \to l \to j$ *is given by* $q^*_{ij/kl} = c_{ij/kl} + \frac{1}{\Theta}\left(1 + W_0\left(\frac{1}{\eta_{ij}} \sum_{(s,t)\in H_{ij}^{F*}} e^{-\Theta c_{ij/st} - 1}\right)\right)$, *where* W_0 *is the principal branch of the Lambert W function.*

One could ask if a similar result holds for the Leader? How many equilibria are there? Are they finite? But before we give some answers, we are going to prove one small lemma.

Lemma 1. *If the Follower uses a fixed margin in his best response on all OD pairs, then the Leader should also use a fixed margin in his best response.*

Proof. It is enough to prove that First Order Conditions (FOC) are satisfied only for a fixed margin. Objective function is decomposable, so we can focus our attention to some particular OD pair $(i,j) \in N^2$, thus neglecting the OD indices. This reduces analysis to the objective

$$\max w \frac{\sum\limits_{(k,l)\in H^L} (p_{kl} - c_{kl}) e^{-\Theta p_{kl}}}{\sum\limits_{(k,l)\in H^L} e^{-\Theta p_{kl}} + \gamma^*}$$

In next few lines, we give a sketch for the essentially straightforward proof. For particular hubs s and t, we can compute $\frac{\partial \gamma^*}{\partial p_{st}}$ from the FOC expression. The derivative can also be computed from (4), using the Theorem 1. These two expression are constructing an equation, hard-wired to the hubs s and t. We can derive the similar equation choosing some other hubs, e.g. $(m,g) \in H^L$, and from there to obtain that $p_{st} - c_{st} = p_{mg} - c_{mg}$. □

Now, we present the theorem about the Stackelberg pricing equilibrium.

Theorem 2. *In LFHLPP, where hub and spoke networks are already given, there is a unique finite Stackelberg equilibrium in terms of pricing.*

Proof. Like in the preceding lemma, we focus our attention to OD pairs. Using the definition of the Lambert W function, the subject of our analysis becomes

$$z(r) = \frac{wr}{1 + W_0(Qe^{\Theta r - 1})}$$

where $r = p_{kl} - c_{kl}$, and $Q = \sum_{(k,l)\in H^{F*}} e^{-\Theta c_{kl}} / \sum_{(k,l)\in H^L} e^{-\Theta c_{kl}}$ (which is always greater than 0). Now, it is easy to see that $z(r)$ has a unique maximizer r^*. FOC can be written as a system

$$x^2 + (2-\Theta r)x + 1 = 0 \tag{16}$$

$$x = W_0(Qe^{\Theta r - 1}) \tag{17}$$

The quadratic equation (16) gives us that the feasible solution exists only when $r \geq \frac{4}{\Theta}$. From the straightforward examination of the slopes for the left and right hand sides of (17), we can conclude that $z(r)$ has only one maximum. □

The number of possible hub and spoke networks for both players is finite. For each pair of them there is a Stackelberg pricing equilibrium. So, for both types of the Follower's behaviour, there exists a finite optimal solution for the Leader.

Theorem 3. *A Stackelberg equilibrium exists in the LFHLPP.*

4 Nash Pricing Equilibrium

One can think about relaxing the pre-commitment in terms of pricing, a.k.a "the price war" from [6]. Thus, we need to show if the Nash equilibrium exists.

Theorem 4. *For already given hub and spoke networks there is a unique finite Nash equilibrium in terms of pricing.*

Proof. Again, we can focus our attention to the OD pairs, and neglect the corresponding indices. For both competitors we have the expressions for their best responses, that is $r_L(r_F) = \frac{1}{\Theta}\left(1 + W_0\left(Qe^{\Theta r_F - 1}\right)\right)$ and $r_F(r_L) = \frac{1}{\Theta}\left(1 + W_0\left(\frac{e^{\Theta r_L - 1}}{Q}\right)\right)$ for the Leader and the Follower, respectively. Here, $Q = \frac{\sum_{(k,l)\in H^L} e^{-\Theta c_{kl}}}{\sum_{(k,l)\in H^F} e^{-\Theta c_{kl}}}$. Now, the equation $r_L^* = r_L(r_F^*) = r_L(r_F(r_L^*))$ needs to be solved, which is equivalent to

$$t = W_0\left(Qe^{W_0\left(\frac{e^t}{Q}\right)}\right) \tag{18}$$

$$r_L^* = \frac{t+1}{\Theta} \tag{19}$$

Taking into account that $W_0(Qe^{W_0(\frac{e^t}{Q})}) = Qe^{W_0(\frac{e^t}{Q})}$, we can transform (18)-(19) into

$$W_0(Qe^{\xi}) = \frac{1}{\xi} \ \wedge \ \xi > 0 \tag{20}$$

$$\xi = W_0\left(\frac{e^t}{Q}\right) \tag{21}$$

$$r_L^* = \frac{t+1}{\Theta} \ \wedge \ r_L^* \geq 0 \tag{22}$$

The first equation always has a solution on $(0, \infty)$. Now, we check the feasibility of the solution, that is if $r_L^* \geq 0$. The last two equations result in

$$e^t = Q\xi e^{\xi} \, \wedge \, \xi > 0 \, \wedge \, t \geq -1 \quad \Leftrightarrow \quad \xi \geq W_0\left(\frac{1}{Qe}\right)$$

What is left to be shown is that $W_0(Qe^{W_0((Qe)^{-1})}) \leq 1/W_0((Qe)^{-1})$, for all $Q > 0$. For that one could just analyse a function $f(Q) = W_0((Qe)^{-1})W_0(Qe^{W_0((Qe)^{-1})})$ on the corresponding interval. □

When it comes to the profit, Stackelberg pricing equilibria is the best one-shot move, by its concept. But these two scenarios, could lead to different outcomes from the hub location point of view. It is not clear that the scenario with the pre-commitment in terms of pricing (LFHLPP) will bring more profit. Nevertheless, we have the following conclusion.

Theorem 5. *In the leader-follower hub location and pricing competition, where competitors are allowed to change their prices, there is a profit maximising solution for the Leader.*

5 Conclusion and Future Work

We have analysed the Leader-Follower setting for hub location and pricing problem, extending the results of Lüer-Villagra and Marianov [3]. It is shown that, when it comes to the pricing, there is a unique solution for the Leader to minimize the damage that can be done by the Follower. This result implied the existence of the solution for this problem.

In future, we plan to address this problem from the computational point of view, and to compare the solutions for LFHLPP and the ”pricing war” version. Another line of the research is oriented to a setting where the demand is elastic.

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