LOWER BOUNDS FOR THE UNCAPACITATED FACILITY LOCATION PROBLEM WITH USER PREFERENCES

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Abstract. We consider the bilevel uncapacitated facility location problem with user preferences. It is known that this model may be reformulated as a one-level location problem with some additional constraints. In this paper we introduce a new reformulation and show that this reformulation dominates three previous ones from the point of view of their linear programming relaxations and may be worse than a reduction to the row selection problem for pairs of matrices. However, this last reduction requires many additional variables and constraints. Computational experiments on random data instances shows that the new reformulation allows to find an optimal solution of the bilevel location problem considered faster than all previous approaches.

Key words. Bilevel programming, facility location, pseudo-Boolean function, duality gap.

1 Introduction

In hierarchical bilevel mathematical models we have two decision makers. One of them is called the *leader*. The other one is called the *follower*. Both decision makers have their own objective function and variables. For a given value of the leader's variables the follower solves his optimization problem. The optimal solution of the follower allows the leader to compute his objective function's value. The main purpose in the bilevel problem is to optimize the leader's objective function. Models of this type are known as Stackelberg games [11].

In this paper we consider the bilevel uncapacitated facility location problem with user preferences. The leader is a production company. The follower is a user or set of users. Polynomially solvable cases, complexity results, reductions to the minimization of pseudo-Boolean functions and reformulations of the problem as a one-level location problem can be found in [8, 9]. We propose a new reformulation and study the relationship with previous ones. It is shown that the new reformulation dominates three previous reformulations from the point of view of their linear programming relaxation and may be worse than the reduction to the row selection problem for pairs of matrices [4]. This last reduction can improve the lower bound for the branch and bound method but requires many additional constraints and variables. Computational experiments on random data instances indicates the superiority of the new reformulation. The commercial software CPLEX requires less efforts to find an optimal solution with the new reformulation than in all previous approaches.

The paper is organized as follows. In Section 2 we present the mathematical formulation of the Uncapacitated Facility Location Problem with User Preferences (UFLPUP). Section 3 contains three reformulations of this problem, proposed in [8], and a new reformulation. Section 4 is devoted to the duality gap. Reductions of UFLPUP to the minimization problem for pseudo-Boolean function and to the row selection problem for pairs of matrices are described in Sections 5 and 6. Computational experiments are discussed in the final Section 7.

2 Problem formulation

Consider a set of facilities $I = \{1, \dots, m\}$ and a set of users $J = \{1, \dots, n\}$. For the company, we are given the fixed costs of facilities $f_i \geq 0, i \in I$ and transportation costs $c_{ij}, i \in I, j \in J$. For users, we are given preferences $d_{ij}, i \in I, j \in J$.

Problem variables are:

$$y_i = \begin{cases} 1 & \text{if a facility } i \text{ is opened,} \\ 0 & \text{otherwise,} \end{cases}$$

$$x_{ij} = \begin{cases} 1 & \text{if user } j \text{ is served from facility } i, \\ 0 & \text{otherwise.} \end{cases}$$

The UFLPUP can be written as a 0-1 program [8]:

$$\min_{y_i} \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}(y) \tag{1}$$

$$s.t. \qquad y_i \in \{0, 1\}, \qquad i \in I, \tag{2}$$

where $x_{ij}(y)$ is an optimal solution of the following inner problem:

$$\min_{x_{ij}} \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} \tag{3}$$

$$s.t. \qquad \sum_{i \in I} x_{ij} = 1, \qquad j \in J \tag{4}$$

$$0 \leq x_{ij} \leq y_i, \qquad i \in I; j \in J. \tag{5}$$

If $d_{ij} = c_{ij}, i \in I, j \in J$ we get the well-known single level Uncapacitated Facility Location Problem (UFLP) which can be written as (1), (2), (4), (5).

We suppose that the optimal solution of the inner problem is unique for any arbitrary solution y. Otherwise this bilevel problem is not well defined. If we allow different optimal solutions for the same y then the definition of the objective function (1) is not correct. For simplicity, we assume $d_{ij} \neq d_{kj}$ for $i, k \in I,\ i \neq k$, and $j \in J$. In the general case, we have to consider optimistic and pessimistic evaluations of the total cost for the company or(and) introduce additional assumptions concerning user behavior. For practical purposes we may assume that all values of d_{ij} are different for each $j \in J$.

3 Reformulations

Observe that only the ranking of the d_{ij} for each j is of importance and not their numerical values. Let the ranking for user $j \in J$ be

$$d_{i_1 j} < d_{i_2 j} < \dots < d_{i_m j}, \tag{6}$$

and $S_{ij} = \{k \in I \mid d_{kj} < d_{ij}\}$, $T_{ij} = \{k \in I \mid d_{kj} > d_{ij}\}$ for all $i \in I$. For an optimal solution of the inner problem we have

$$x_{ij} = 1 \Longrightarrow y_k = 0,\ k \in S_{ij}. \tag{7}$$

We can therefore re-write UFLPUP as follows:

$$\min_{y_i, x_{ij}} \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \tag{8}$$

$$\text{s.t.} \qquad x_{ij} + y_k \leq 1, \qquad k \in S_{ij}; \quad i \in I; j \in J \tag{9}$$

$$\sum_{i \in I} x_{ij} = 1, \qquad j \in J \tag{10}$$

$$0 \leq x_{ij} \leq y_i, \quad i \in I; j \in J \tag{11}$$

$$y_i \in \{0, 1\}, \qquad i \in I. \tag{12}$$

Indeed, in every optimal solution of (8) - (12) all constraints of UFLP will be satisfied, and constraints (9) will ensure that x_{ij} is an optimal solution for the inner problem. The number of variables of problem (8) - (12) is $m+mn$, as in the usual UFLP. However, while the UFLP has the already large number of constraints $n + mn$, problem (8) - (12) has $O(m^2 n)$ additional ones. This prohibits a direct resolution except for small instances. To avoid too numerous additional constraints we can re-write (7) in the equivalent form:

$$\sum_{k \in S_{ij}} y_k \leq |S_{ij}|(1 - x_{ij}), \qquad i \in I; j \in J. \tag{13}$$

So, we have the same number of variables and mn additional constraints only. The new constraints (13) are obtained by summing the constraints (9). We have got the same integer programming problem but the linear programming relaxation is weaker in this case. To improve this relaxation we can re-write (7) as follows:

$$y_i \leq x_{ij} + \sum_{k \in S_{ij}} y_k \qquad i \in I; j \in J. \tag{14}$$

These three reformulations are suggested in [8]. Our first result deals with a new reformulation of UFLPUP which provides a better linear programming relaxation than the three previous ones. Let us re-write (7) in the following way:

$$y_i \leq x_{ij} + \sum_{k \in S_{ij}} x_{kj} \qquad i \in I; j \in J. \tag{15}$$

Theorem 3.1 The optimal value of the linear programming relaxation for (8), (10)–(12), (15) is greater than or equal to the corresponding value for each of the previous three reformulations.

Proof. From (10) we have

$$1 - \sum_{k\in T_{ij}} x_{kj} = x_{ij} + \sum_{k\in S_{ij}} x_{kj}.$$

If we replace the right-hand side of (15) then we get $y_i \le 1 - \sum_{k\in T_{ij}} x_{kj}$ which implies (9). Other statements are obvious. □

4 Duality gap

Let us consider the special case of UFLPUP when $f_i = 0, i \in I$. We will show that the duality gap for this case may be close to 1. Moreover, this *simple* case is NP-hard and equivalent to the general case.

Theorem 4.1 There is a family of data instances for UFLPUP such that the duality gap is arbitrary close to 1 even for $f_i = 0, i \in I$.

Proof. Put $I = \{1, \dots, k, k+1\}, J = \{1, 2\}$ and

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} k+1 & k+1 \\ k & k \\ \vdots & \vdots \\ 1 & 1 \end{pmatrix}$$

For any arbitrary nonempty set $S \subseteq I$ of facilities the value of objective function (8) is equal to 1, $F_{IP}(S) = 1$. So, the optimal value $F^*_{IP} = 1$. In order to find an optimal value F^*_{LP} for the linear programming relaxation of (8), (10)- (12), (15) we consider the dual problem:

$$\max_{v_j, w_{ij}, u_{ij}} \sum_{j\in J} v_j$$

$$s.t.\ v_j \le c_{ij} + w_{ij} - u_{ij} - \sum_{k\in T_{ij}} u_{kj}, \quad i \in I; j \in J,$$

$$\sum_{j\in J} w_{ij} \le \sum_{j\in J} u_{ij}, \quad i \in I,$$

$$w_{ij} \ge 0, u_{ij} \ge 0, \quad i \in I; j \in J.$$

We wish to verify that the following dual solution $v = (0, 1/k)$,

$$W = \begin{pmatrix} 0 & 0 \\ 0 & 1/k \\ \vdots & \vdots \\ 0 & 1/k \end{pmatrix} \quad U = \begin{pmatrix} 0 & 0 \\ 1/k & 0 \\ \vdots & \vdots \\ 1/k & 0 \end{pmatrix}$$

and the primal solution

$$Y = \begin{pmatrix} 1 \\ 1/k \\ \vdots \\ 1/k \end{pmatrix} \qquad X = \begin{pmatrix} 1-1/k & 0 \\ 0 & 1/k \\ \vdots & \vdots \\ 0 & 1/k \\ 1/k & 1/k \end{pmatrix}$$

are optimal. Both solutions are feasible and their objective functions have the same value, $F^*_{LP} = 1/k$. Hence, the duality gap is:

$$(F^*_{IP} - F^*_{LP})/F^*_{IP} = 1 - 1/k,$$

which increases with k and goes to 1 in the limit. □

A similar result is known for UFLP [10], but we assume here that $f_i = 0, i \in I$. For UFLP this is a trivial case. So, taking into account user preferences makes the problem more sophisticated indeed.

Theorem 4.2 [8] UFLPUP and its special case for $f_i = 0, i \in I$ are equivalent.

To reduce UFLPUP to the special case we introduce m additional users with the following transportation costs and preferences:

$$C = \begin{pmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_n \end{pmatrix} \quad D = \begin{pmatrix} 1 & & >1 \\ & \ddots & \\ >1 & & 1 \end{pmatrix}$$

This new problem, with additional users, has the same objective function value as the UFLP for any nonempty set of facilities. So, the problems are equivalent. □

Corollary 4.1 UFLPUP is NP-hard in the strong sense even for $f_i = 0, i \in I$.

5 Pseudo-Boolean polynomials

Peter Hammer was the first to propose a reduction of UFLP to the minimization problem for pseudo-Boolean function [6, 7]. Later, Vladimir Beresnev suggested another reduction, which gives a simple way to get a corresponding UFLP for a given pseudo-Boolean function and vice-versa [2, 3]. The reduction of Beresnev is elegant and easy to understand. It is based on the following observation ([3], Lemma 1.1).

For a given vector $g_i, i \in I$ with ranking

$$g_{i_1} \le g_{i_2} \le \cdots \le g_{i_m},$$

we introduce a vector $\Delta g_i, i = 0, \ldots, m-1$ in the following way:

$$\Delta g_0 = g_{i_1},$$

$$\Delta g_l = g_{i_{l+1}} - g_{i_l}, \quad 1 < l < m.$$

For any arbitrary vector $z_i \in \{0,1\}, i \in I, \ z \neq (1, \ldots, 1)$, the following statement holds:

$$\min_{i | z_i = 0} g_i = \sum_{l=0}^{m-1} \Delta g_l z_{i_1} \ldots z_{i_l}.$$

Using this equation, one can get a pseudo-Boolean function for UFLP:

$$p(z) = \sum_{i \in I} f_i (1 - z_i) + \sum_{j \in J} \sum_{l=0}^{m-1} \Delta c_{lj} z_{i_1^j} \ldots z_{i_l^j}. \quad (16)$$

The ranking $i_i^j, \ldots, i_m^j$ is generated by the column j of the matrix C:

$$c_{i_1^j} \le c_{i_2^j} \le \cdots \le c_{i_m^j}, \quad j \in J.$$

An optimal solution $z_i^*, i \in I$ for the minimization problem for this pseudo-Boolean function with restriction $z \neq (1, \ldots, 1)$ gives us an optimal solution for UFLP, $y_i^* = 1 - z_i^*, i \in I$ and vice-versa ([3], Theorem 3.2). V. Beresnev used this statement to enlarge the set of known polynomially solvable cases for UFLP. In fact, if two instances can be reduced to the same pseudo-Boolean function and one of them is easy to solve then both instances are easy to solve. In this sense, the function $p(z)$ is something like a kernel of UFLP.

Note that $p(z)$ has no negative nonlinear terms. In [8] it is shown that UFLPUP is equivalent to the minimization problem for a pseudo-Boolean function. The function may have negative nonlinear terms. The reduction is similar to the previous one but uses the ranking (6) instead of the ranking used in (16). More exactly, for ranking (6) we define

$$\nabla c_{i_1 j} = c_{i_1 j}, \quad \nabla c_{i_l j} = c_{i_l j} - c_{i_{l-1} j}, \ 1 < l \le m,$$

and consider the minimization problem for the pseudo-Boolean function (PBFP):

minimize

$$P(z) = \sum_{i \in I} f_i (1 - z_i) + \sum_{j \in J} \sum_{i \in I} \nabla c_{ij} \prod_{k \in S_{ij}} z_k, \quad (17)$$

$$\text{s.t. } z \neq (1, \ldots, 1).$$

Theorem 5.1 [8] PBFP and UFLPUP are equivalent.

Notice that the coefficients ∇c_{ij} may be positive or negative. In other words, for any pseudo-Boolean function we may design an equivalent data instance of UFLPUP and vice-versa. Hence, we don't need to separate the first term in (17) and we get another proof of Theorem 4.2 in terms of pseudo-Boolean functions. Further, it will be convenient to use $P(z)$ in the following form:

$$P(z) = -\sum_{j \in J^-} a_j \prod_{i \in \alpha_j} z_i + \sum_{j \in J^+} b_j \prod_{i \in \beta_j} z_i, \quad (18)$$

where $a_j > 0, j \in J^-, b_j > 0, j \in J^+, \alpha_j, \beta_j \subseteq I$, and $J^- \cup J^+$ is the set of terms for $P(z)$.

Example 1. For the data instance of UFLPUP in section 4, we have

$$\nabla c_{ij} = \begin{cases} -1 & \text{if } j = 1, i = 1 \\ +1 & \text{if } j = 1, i = k+1 \text{ or } j = 2, i = 1 \\ 0 & \text{otherwise} \end{cases}$$

and $P(z_1, \ldots, z_{k+1}) = 1 + z_2 \ldots z_{k+1} - z_2 \ldots z_{k+1} = 1$. Recall $F_{IP}(S) = 1$, for all $S \subseteq I, S \neq \emptyset$.

6 The row selection problem for pairs of matrices

Let us consider a pair of matrices $A = (a_{ij}), i \in I, j \in J_1$ and $B = (b_{ij}), i \in I, j \in J_2$ which have the same number of rows and maybe different numbers of columns. The row selection problem for pairs of matrices (PMP) is to find a nonempty set of rows $S \subseteq I$ which minimizes the objective function:

$$R(S) = \sum_{j \in J_1} \max_{i \in S} a_{ij} + \sum_{j \in J_2} \min_{i \in S} b_{ij}.$$

If $J_1 = I$ and $a_{ij} = f_i$ for $i = j$ and 0 otherwise, we get UFLP. So, it is an NP-hard problem in the strong sense.

Theorem 6.1 (Beresnev) [4] PMP and PBFP are equivalent.

But PBFP is equivalent to UFLPUP. Hence, we may re-write UFLPUP as PMP in order to get a lower bound [8]. For (18) put $J_1 = J^-, J_2 = J^+$ and

$$a_{ij} = \begin{cases} a_j & \text{if } i \in \alpha_j \\ 0 & \text{otherwise} \end{cases} \quad j \in J_1,$$

$$b_{ij} = \begin{cases} 0 & \text{if } i \in \beta_j \\ b_j & \text{otherwise} \end{cases} \quad j \in J_2.$$

We have

$$P(z) = R(S) - \sum_{j\in J^-} a_j.$$

To obtain a new lower bound we will use an integer program for PMP and its LP-relaxation. Note that every column of matrix A has two different values only. So, we can write PMP as follows:

$$\min_{t_j, x_{ij}} \sum_{j\in J_1} a_j t_j + \sum_{j\in J_2}\sum_{i\in I} b_{ij}x_{ij} \tag{19}$$

$$\text{s.t.} \quad \sum_{i\in I} x_{ij} = 1,\ j\in J_2 \tag{20}$$

$$t_{j_1} \geq \sum_{i\in \alpha_{j_1}} x_{ij},\ j_1\in J_1,\ j\in J_2 \tag{21}$$

$$t_{j_1}, x_{ij} \in \{0,1\},\ j_1\in J_1, i\in I, j\in J_2. \tag{22}$$

The dual problem is the following:

$$DR = \max \sum_{j\in J_2} \overline{v}_j$$

$$\text{s.t.} \quad \overline{v}_j \leq b_{ij} + \sum_{j_1\in J_1 | i\in\alpha_{j_1}} r_{j_1 j}, \quad j\in J_2, i\in I;$$

$$\sum_{j\in J_2} r_{j_1 j} \leq a_{j_1}, \quad j_1\in J_1;$$

$$r_{j_1 j}\geq 0,\ j_1\in J_1, j\in J_2.$$

A new lower bound is:

$$LB = DR - \sum_{j\in J_1} a_j \leq \min_{S\subseteq I, S\neq\emptyset} R(S) - \sum_{j\in J_1} a_j$$

$$= \min_{z\in\{0,1\}, z\neq(1,\dots,1)} P(z) = F^*_{IP}.$$

Theorem 6.2. For arbitrary $N > 0$ there exist a family of data instances for UFLPUP for which $LB \geq NF^*_{LP}$.

Proof. We return to Example 1 and compute the new lower bound. We have $J^- = J^+ = \{1\}, \alpha_1 = \beta_1 = \{2,\dots,k+1\}, a_1 = b_1 = 1$, and

$$A = \begin{pmatrix} 0\\ 1\\ \vdots\\ 1\end{pmatrix} \qquad B = \begin{pmatrix} 1\\ 0\\ \vdots\\ 0\end{pmatrix}$$

An optimal solution is $r_{11} = \overline{v}_1 = 1$ and $LB = \overline{v}_j - a_1 + 1 = 1$. □

7 Computational results

The lower bound LB looks like the best one. But the problem (19)–(22) has $O(nm)$ variables and $O(n^2m)$ constraints. For comparison, the problem (8), (10)–(12), (15) has $O(nm)$ variables and $O(nm)$ constraints. It is not clear which reformulation is better for exact solution methods. To answer the question we use the CPLEX software for these two reformulations. Data instances are generated at random by the following rule. The transportation costs are Euclidean distances between random points on the two-dimension plane and $f_i = \sqrt{n}/10, i\in I$ [1]. For user preferences we put $d_{ij} = c_{ij}, i\in I, j\in J$ and produce some random perturbations for each $j\in J$. Table 1 shows the computational results for three reformulations: Model 1: (8), (10) - (12), (14); Model 2: (8), (10) - (12), (15); Model 3: (19) - (22). Parameters *Rows* and *Columns* indicate the dimension of the correspondent models for $n = m = 30$. Ten instances were solved each time and average values are reported. Duality gap is presented in the row *Gap*. Parameters *Nodes* and *Iterations* show the number of nodes visited in the branching tree and the total number of simplex iterations. Running time is reported for Sunfire 4800, 4 CPU 900 MHz, 8 Gb RAM computer.

	Model 1	Model 2	Model 3
Rows	1770	1770	12340
Columns	900	900	1352
Gap	20,98 %	19,26 %	6,33 %
Nodes	439	94	27
Iterations	15359	4567,8	32874
CPU Time	20,05	6,50	889,40

Table 1. Average performance of CPLEX

Model 3 has a small duality gap and a small number of visited nodes in the branching tree. However, the running time is high. Model 2 requires the smallest number of simplex iterations and the smallest running time. It is interesting to note that Model 1 has a slightly larger duality gap and substantially higher running time, number of nodes, and iterations for the same dimensions. In further research it will be interesting to develop local search heuristics in order to start CPLEX from a good or an optimal solution.

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