

Complexity of local search for the p -median problem *

Ekaterina Alekseeva, Yuri Kochetov, Alexander Plyasunov

Sobolev Institute of Mathematics, Novosibirsk
{ekaterina2, jkochet, apljas}@math.nsc.ru

Abstract

For the p -median problem we study complexity of finding a local minimum in the worst and the average cases. We introduce several neighborhoods and show that the corresponding local search problems are PLS-complete. In the average case we note that standard local descent algorithm is polynomial. A relationship between local optima and 0–1 local saddle points is presented.

Keywords: Local descent, PLS-completeness, pivoting rules, Karush-Kuhn-Tucker conditions.

1 Introduction

In the p -median problem we are given a set $I = \{1, \dots, n\}$ of potential locations for p facilities, a set $J = \{1, \dots, m\}$ of customers, and a matrix (g_{ij}) , $i \in I$, $j \in J$ of transportation costs to serve the customers from the facilities. The goal is to find a subset $S \subset I$, $|S| = p$ such that to minimize the objective function

$$F(S) = \sum_{j \in J} \min_{i \in S} g_{ij}.$$

It is well-known combinatorial problem [5] which is NP-hard in the strong sense.

In this paper we study complexity of finding a local minimum for polynomially searchable neighborhoods. We present a sufficient condition when this local search problem is PLS-complete. Several polynomial neighborhoods are introduced and it is shown that in the worst case the standard local descent algorithm takes exponential number of steps with each neighborhood regardless of the tie-breaking and pivoting rules used. But in the average case this algorithm is probably polynomial.

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We consider some pivoting rules and present computational results for random test instances. We note that the number of steps for the algorithm grows as a linear function for the pivoting rules *Best improvement* and *First improvement* and grows like a quadratic function for the *Worst improvement* rule. Some theoretical explanations of this phenomenon are discussed.

The paper is organized as follows. In Section 2 we define some neighborhoods. In Section 3 the PLS-completeness of the p -median problem with these neighborhoods is established. We define approximate local optima and study corresponding local search problems. In Section 4 a relationship between Swap local optima and 0–1 local saddle points is presented. We rewrite the p -median problem as a minimization problem for pseudo-Boolean function and show that Swap local optima are 0–1 vectors satisfied the Karush–Kuhn–Tucker conditions and vice versa. In Section 5 we study the running time of the local descent algorithms in the average case. Pivoting rules are introduced in Section 6. Computational results and further research directions are discussed in Sections 7, 8.

2 Neighborhoods

The *Swap* neighborhood is one of the effective and efficient neighborhoods for the p -median problem [7]. It contains all subsets $S' \subset I$, $|S'| = p$, with the Hamming distance from S' to S at most 2. Similarly, the k -Swap neighborhood is the set of all feasible solutions with Hamming distance from S' to S at most k . Finding the best element in this neighborhood is time consuming for large k . So, this neighborhood is interesting for theoretical study only.

The Lin-Kernighan neighborhood (LK) is a subset of the k -Swap neighborhood. It consists of k elements, $k \leq n - p$, and can be described by the following steps [4].

Step 1. Choose two facilities $i_{ins} \in I \setminus S$ and $i_{rem} \in S$ such that $F(S \cup \{i_{ins}\} \setminus \{i_{rem}\})$ is minimal even if it greater than $F(S)$.

Step 2. Perform exchange of i_{rem} and i_{ins} .

Step 3. Repeat k times the steps 1,2 so that a facility can not be chosen to be inserted in S if it has been removed from it in one of the previous iterations of step 1 and step 2.

The sequence $\{(i_{ins}^\tau, i_{rem}^\tau)\}_{\tau \leq k}$ defines k neighborhoods S_τ for solution S . We say that S is a local minimum with respect to LK -neighborhood if $F(S) \leq F(S_\tau)$ for all $\tau \leq k$. Let us define the neighborhood $LK_1(S)$ as a subset of $LK(S)$ which contains one element only, $S_\tau, \tau = 1$. By

definition, $LK_1(S) \subset Swap(S)$.

The Fiduccia-Mattheyses neighborhood (FM) is defined as the LK -neighborhood with a another rule for the choice of facilities i_{ins} and i_{rem} at the step 1 [2]. Now, this step consists of two stages. At the first stage we select $i_{rem} \in S$ such that $F(S \setminus \{i_{rem}\})$ is minimal. At the second stage we find $i_{ins} \in I \setminus S$ such that $F(S \cap \{i_{ins}\} \setminus \{i_{rem}\})$ is minimal. It defines the sequence S_τ , $\tau \leq k$ of neighbors for the solution S . Neighborhood $FM_1(S)$ contains only the first element from this sequence.

We say that neighborhood N_1 is stronger than neighborhood N_2 ($N_2 \preceq N_1$) if every N_1 -local minimum is N_2 -local minimum. It is early to verify that

$$\begin{aligned} FM_1 &\preceq Swap \preceq LK_1 \preceq LK, \\ LK_1 &\preceq Swap \preceq k - Swap, \\ FM_1 &\preceq FM. \end{aligned}$$

For a given constant $k > 0$ all neighborhoods are polynomial. Hence, the p -median problem with every of these neighborhoods belongs to the class PLS [9].

3 The worst case complexity

There are the most difficult local search problems in the class PLS. They are called PLS-complete problems [3]. One of them is the graph bipartition problem with FM_1 neighborhood [9]. We claim that the p -median problem with FM_1 neighborhood is the most difficult local search problem as well.

Theorem 1. *The p -median problem with FM_1 neighborhood is PLS-complete.*

Theorem 2. *Suppose that the neighborhood N is stronger than neighborhood FM_1 and the p -median local search problem with neighborhood N belongs to the class PLS. Then it is PLS-complete.*

The standard local descent algorithm starts from an initial solution and moves to a better neighboring solution until terminates at a local minimum.

Theorem 3. *The standard local descent algorithm takes exponential time in the worst case for the $Swap, LK, LK_1, FM, FM_1$ neighborhoods regardless of the tie-breaking and pivoting rules used.*

The standard local minimum problem is the following. We are given an instance of the p -median problem, a neighborhood, and an initial solution. The goal is to find a local minimum with respect

to the neighborhood that would be produced by the standard local descent algorithm starting from the initial solution.

Theorem 4. *Standard local minimum problem for the Swap, LK, LK₁, FM, FM₁ neighborhoods is PSPACE-complete.*

For $\varepsilon > 0$ solution S^ε is called a (ε, N) -local minimum if $F(S^\varepsilon) \leq (1 + \varepsilon)F(S)$ for all $S \in N(S^\varepsilon)$.

Theorem 5. *If the neighborhood N is polynomially searchable then a (ε, N) local minimum for the p -median problem can be founded in polynomial time both in the problem size and $1/\varepsilon$.*

Theorem 6. *If $FM_1 \preceq N$ and there is a polynomial time algorithm to find a feasible solution S^0 for the p -median problem such that $F(S^0) \leq F(S) + \varepsilon$ for all $S \in N(S^0)$ and a constant $\varepsilon > 0$ then we can find a local optimum in polynomial time for all problems in the class PLS.*

4 Local saddle points

In this section we show that there is a strong connection between *Swap*-local minimum and the local saddle points for the Lagrange function. Let us rewrite the p -median problem as the minimization problem for a pseudo-Boolean function [1]:

$$P(y) = \sum_{j \in J'} a_j \prod_{i \in I_j} y_i \rightarrow \min$$

$$\text{s.t. } \sum_{i \in I} y_i = n - p, \quad y_i \in \{0, 1\}, \quad i \in I,$$

where $a_j \geq 0, I_j \subset I, j \in J' = \{1, \dots, n \times m\}$. For feasible solution S of the p -median problem we have $i \in S \Leftrightarrow y_i = 0$. Moreover, $F(S) = P(y)$. The Lagrange function with multipliers $\lambda, \mu_i \geq 0, \sigma_i \geq 0, i \in I$ and continuous variables $0 \leq y_i \leq 1, i \in I$ is the following

$$L(y, \lambda, \mu, \sigma) = P(y) + \lambda(\sum_{i \in I} y_i - n + p) + \sum_{i \in I} \sigma_i(y_i - 1) - \sum_{i \in I} \mu_i y_i.$$

The vector $(y^*, \lambda^*, \mu^*, \sigma^*)$ is called the saddle point with respect to *Swap* neighborhood if

$$L(y^*, \lambda, \mu, \sigma) \leq L(y^*, \lambda^*, \mu^*, \sigma^*) \leq L(y, \lambda^*, \mu^*, \sigma^*)$$

for all $\lambda, \mu \geq 0, \sigma \geq 0$ and $y \in \text{Swap}(y^*)$.

Theorem 7. *Boolean vector $y^* = (y_1^*, \dots, y_n^*)$ such that $\sum_{i \in I} y_i^* = n - p$, is a *Swap* local minimum if and only if there are multipliers $\lambda^*, \mu_i^* \geq 0, \sigma_i^* \geq 0, i \in I$ such that vector $(y^*, \lambda^*, \mu^*, \sigma^*)$ is the saddle point of function L with respect to the *Swap* neighborhood.*

Using the Karush–Kuhn–Tucker conditions we can rewrite this statement as follows.

Theorem 8. *Boolean vector $y^* = (y_1^*, \dots, y_m^*)$ is a Swap local minimum if and only if there are multipliers $\lambda^*, \mu_i^* \geq 0, \sigma_i^* \geq 0, i \in I$ such that vector y^* is an optimal solution of the problem:*

$$\begin{aligned} L(y, \lambda^*, \mu^*, \sigma^*) &\longrightarrow \min_{y \in \text{Swap}(y^*)} \\ \text{s.t.} \quad & \left(\sum_{i \in I} y_i - n + p \right) \lambda^* = 0, \\ & \sigma_i^* (y_i - 1) = 0, \quad \mu_i^* y_i = 0, \quad i \in I, \\ & \sum_{i \in I} y_i = n - p. \end{aligned}$$

5 The average case complexity

In section 3 we have shown that in the worst case the standard local descent algorithm takes exponential number of steps to reach local minimum. Now we study the behavior of the algorithm in the average case. Let us assume that we can rank all feasible solutions from the worst solution to the best one. Following [8] we consider the distribution of problem structure rather than the distribution of problem data.

Theorem 9. *Let $F(S)$ be a random function and all rankings of feasible solutions are equally likely to occur. If p is a constant, then the expected number of steps for the standard local descent algorithm with Swap neighborhood is less than $1.5en$ regardless of the tie-breaking and pivoting rules used.*

Theorem 10. *Let $F(S)$ be a random function and all rankings of feasible solution are equally likely to occur. If $p = \lceil \alpha n \rceil$ for given $0 < \alpha < 1$ then the expected number of steps for the standard local descent algorithm with Swap neighborhood is less than $1.5en^2$ regardless of the tie-breaking and pivoting rules used.*

Below we consider several pivoting rules for the Swap neighborhood and present the number of steps for the standard local descent algorithm as a function of n .

6 Pivoting rules

Let $Swap^*(S) = \{S' \in Swap(S) \mid F(S') < F(S)\}$ be a subset of neighbors for S with better value of the objective function. Pivoting rule selects a neighbor for current solution at each step of the local descent. This choice may affect the complexity of the algorithm drastically. We consider six pivoting rules and analysis their influence on the number of steps and relative errors of the local optima obtained. Some of these rules are well known and used in metaheuristics. Other ones are new and help us to understand the landscape of the p -median problem better.

The Best improvement rule selects a solution in $Swap^*(S)$ with the smallest value of the objective function. It seems this rule is the most popular in the local search methods.

The Worst improvement rule selects a solution in $Swap^*(S)$ with the largest value of the objective function. According to this rule we use the most flat direction for descent. So, we guess that this rule produces more steps and the final local minimum may be better than for the previous case.

The Random improvement rule picks a neighbor for S in $Swap^*(S)$ at random. It is one of the fastest pivoting rule and can lead to different local optima from the same starting solution.

The First improvement rule is one of the famous pivoting rule. It prescribe to use an element from $Swap^*(S)$ which is found in $Swap(S)$ at first. We test the neighbors of S in lexicographical order and terminate when the first better neighbor is discovered.

The Circular rule is closely related to the previous one. It differs from it in one point only. The *First improvement* rule begins the search at every step from the same starting position, for example, from the lexicographical minimal position. The *Circular* rule begins from the position where the previous step terminates [6]. An idea of this rule deals with the following observation. In many cases the unprofitable movings for the current solution will be unprofitable for the neighboring solutions. So, it is better to continue the exploring instead of starting from the initial position. However, this property should be checked for each problem particularly.

Finally, the *Maximal Freedom* rule selects a neighbor S' in $Swap^*(S)$ with maximal cardinality of the set $Swap^*(S')$. This rule is more time consuming but it gives us a neighbor with maximal number of directions for further improvement. Number of elements in $Swap^*(S)$ we call the freedom of the solution S . An idea of maximal freedom was to find a rule which generates the maximal number of steps for the local descent algorithm. As we will see later, it is not the case.

7 Computational experiments

We test the local descent algorithm with described pivoting rules on random instances. For all instances we put $n = m$. The values g_{ij} are taken from interval $[0, 1000]$ at random with uniform distribution. We study two cases $p = 15$ and $p = \lceil n/10 \rceil$. The goal of our experiments is to check the conclusions of Theorems 7 and 8 for random matrices (g_{ij}) . Note that we know nothing about the rankings of the feasible solutions.

Figures 2 and 3 show the average number of steps from random starting solution to a *Swap* local optimum. Pivoting rules *Freedom* is presented at the both Figures. For all rules except the *Worst*, the number of steps grows as a linear function. For the *Worst* rule we see nonlinear function. Number of steps for local descent grow rapidly and the difference between the *Worst* and the *Best* rules is extremely high for $n > 100$. So, pivoting rules are important from the viewpoint of running time. Figure 1 confirms the conclusion for relative error as well. The *Best* rule has large average deviation from the best found solution. The *Freedom* rule show the smallest deviation. Note that it is the most time consuming rule. The same behavior of the local descent algorithm we observe for Euclidean instances when the elements g_{ij} are Euclidean distances for randomly chosen points on two dimensional plane.

Figures 5, 6 illustrate the average number of steps for the algorithm in case $p = 15$. All rules show the linear functions for average number of steps. The *Worst* rule has the largest number of steps but its relative error is close to the rules *First*, *Circular*, and *Random*. The *Best* rule leads to the local minima with large relative errors (see Figure 4).

8 Conclusions

For the p -median problem we show that standard local descent algorithm takes exponential number of steps in the worst case and polynomial number of steps in the average case. We introduce several neighborhoods and prove that the corresponding local search problems are PLS-complete. We illustrate the relationship between *Swap* local optima and the classical Karush–Kuhn–Tucker conditions and 0–1 local saddle points. For further research it is interesting to study the distribution of local optima in the feasible domain, to check the *big valley* conjecture, and properties of the basin of attraction for the global optimum.

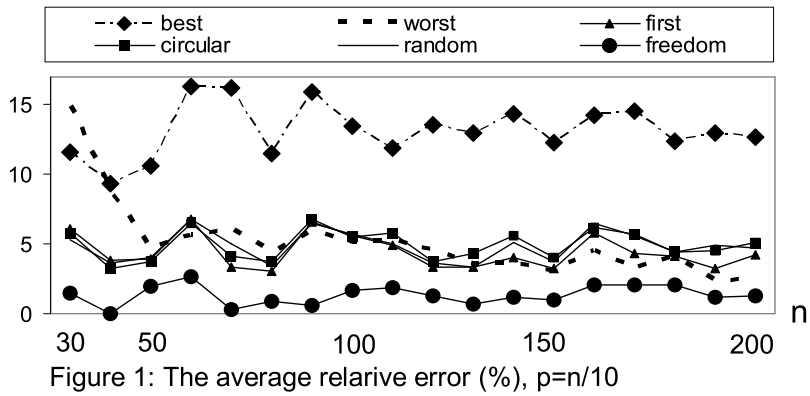


Figure 1: The average relative error (%), $p=n/10$

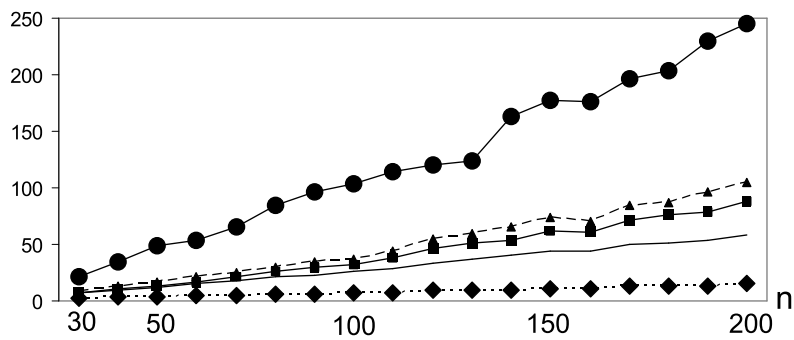


Figure 2: The average number of steps without the *Worst* rule, $p=n/10$

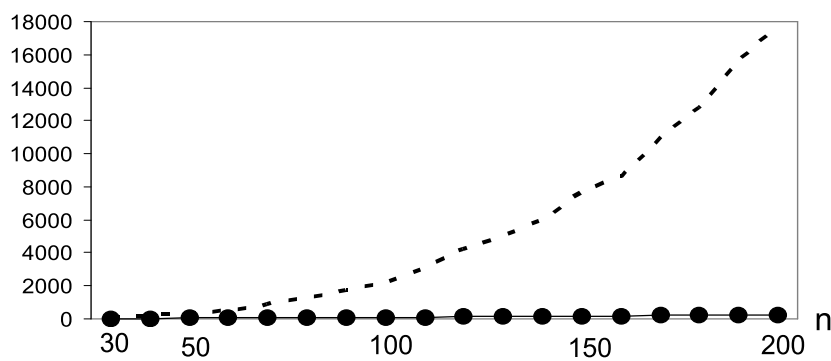


Figure 3: The average number of steps for the *Worst* and the *Freedom* rules, $p=n/10$

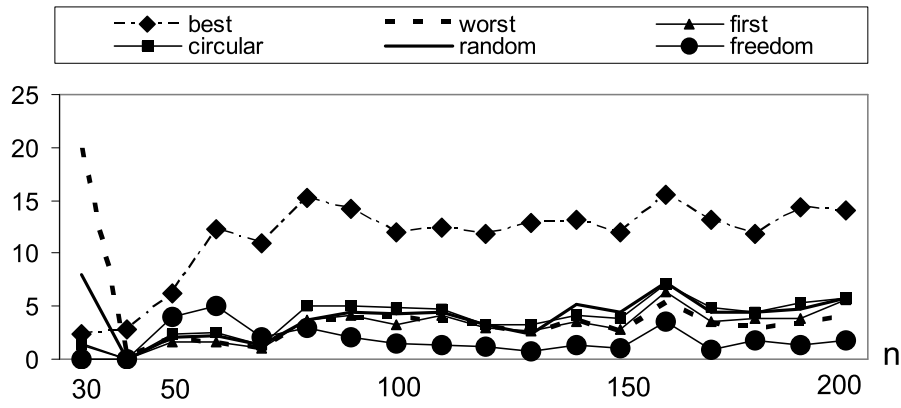


Figure 4: The average relative error (%), $p=15$

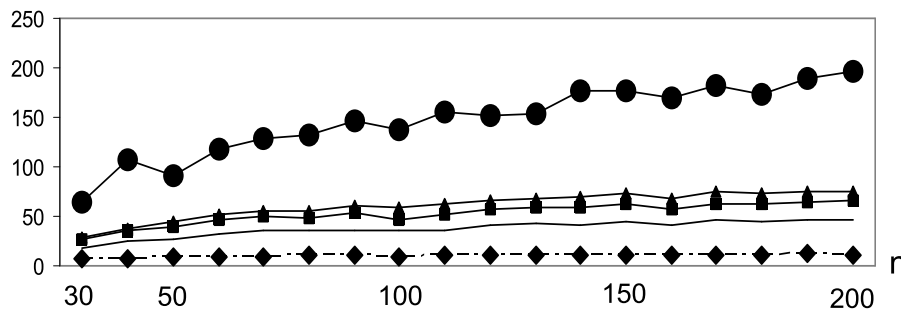


Figure 5: The average number of steps without the *Worst* rule, $p=15$

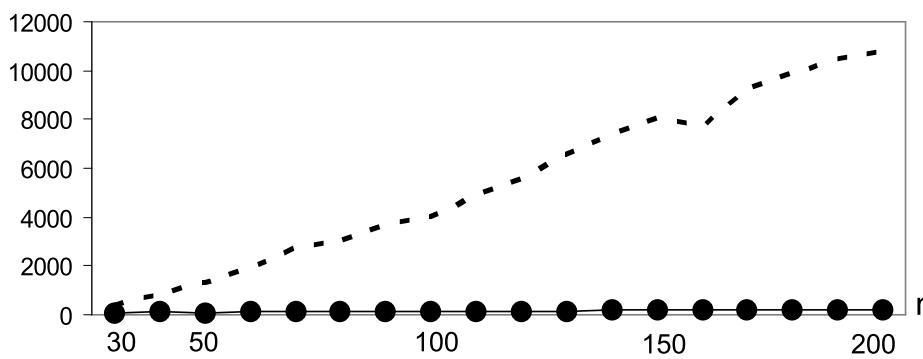


Figure 6: The average number of steps for the *Worst* and the *Freedom* rules, $p=15$

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