Facility location problems
Discrete models and local search methods

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Lecture 1. Facility Location Models

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1. Uncapacitated facility location problem
2. Theoretical and empirical results
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4. Equivalent reformulations and the “best” one
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The Uncapacitated Facility Location Problem

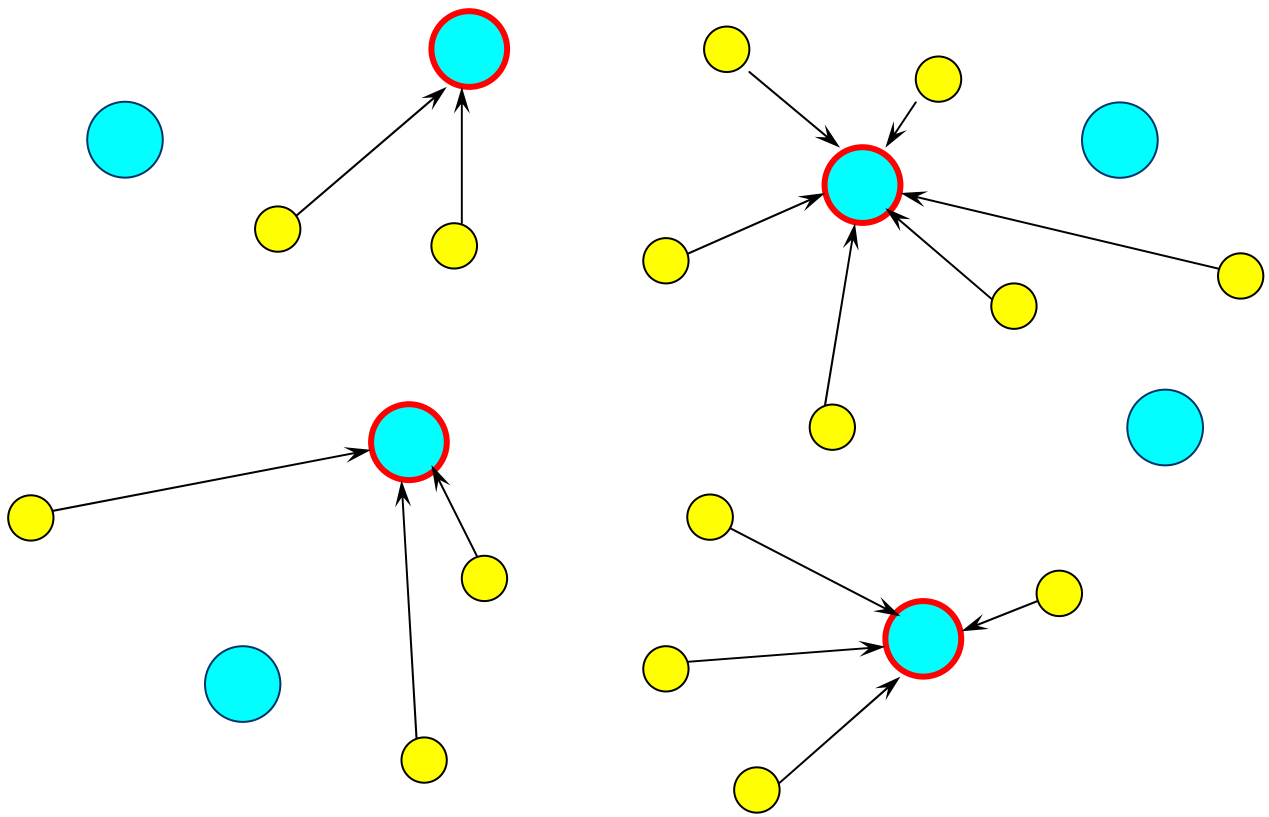
- Input:
  - a set J of users;
  - a set I of potential facilities;
  - a fixed cost f_i of opening facility i;
  - a production-transportation cost c_{ij} to service user j from facility i;
- Output:

  a set $S \subseteq I$ of opening facilities;
- Goal:

  minimize the total cost to open facilities and service all users

$$F(S) = \sum_{i \in S} f_i + \sum_{j \in J} \min_{i \in S} c_{ij} .$$

Example



$I = \{1,\ldots, 8\}$ is potential facility locations;

$J = \{1,\ldots, 15\}$ is set of users

Users are serviced from nearest facility

Integer Programming Formulation

Variables:

$$x_i = \begin{cases} 1 & \text{if facility } i \text{ is open,} \\ 0, & \text{otherwise,} \end{cases}$$

$$y_{ij} = \begin{cases} 1 & \text{if user } j \text{ is served by facility } i, \\ 0, & \text{otherwise,} \end{cases}$$

Mathematical model:

$$\min \left\{ \sum_{i \in I} f_i x_i + \sum_{i \in I} \sum_{j \in J} c_{ij} y_{ij} \right\}$$

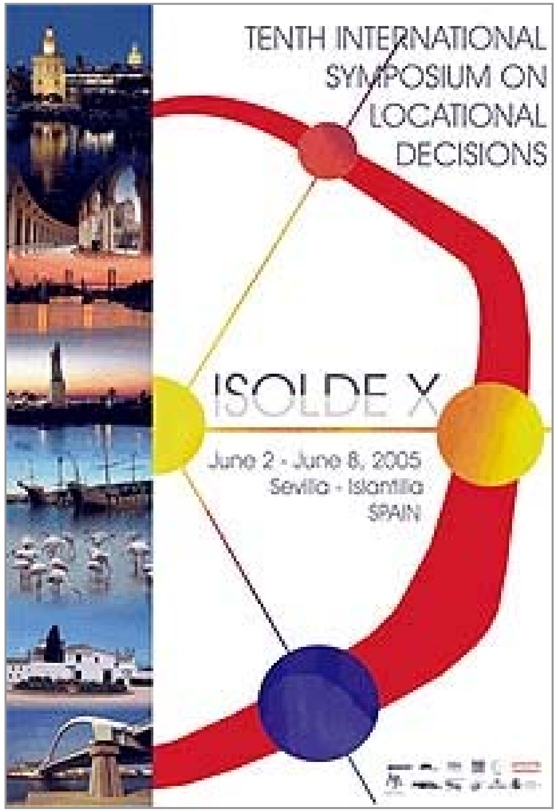
s.t.

$$\sum_{i \in I} y_{ij} = 1, \quad j \in J;$$

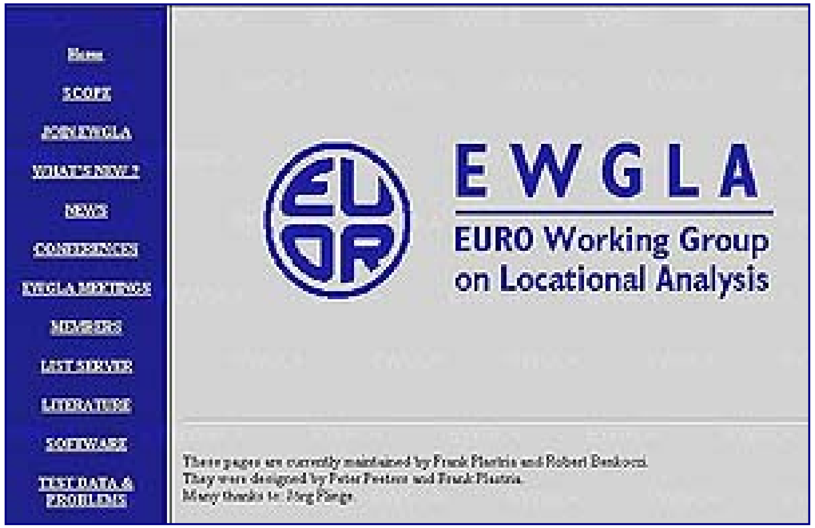
$$x_i \geq y_{ij}, \quad i \in I, j \in J;$$

$$x_i, y_{ij} \in \{0,1\} \quad i \in I, j \in J.$$

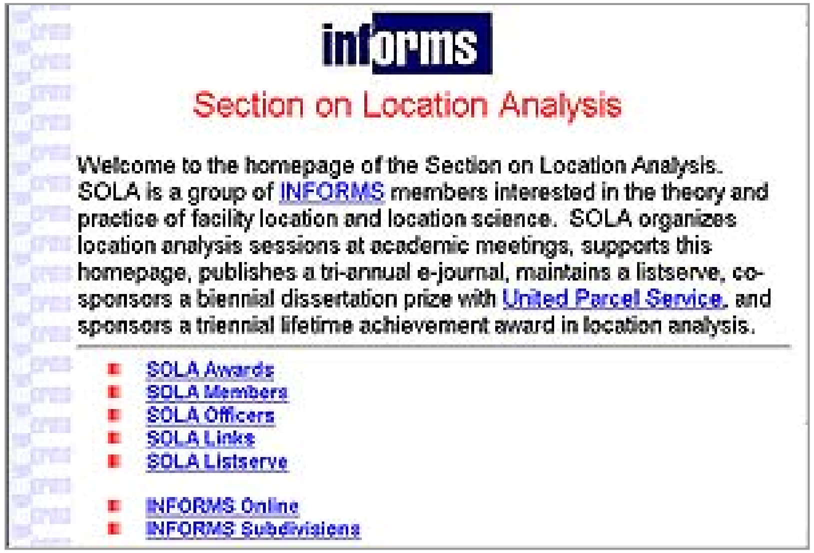
Societies and conferences on Location



International Symposium on Location Decisions
http://www.aloj.us.es/isolde/



Euro Working Group on Locational Analysis
http://www.vub.ac.be/EWGLA/



INFORMS Section on Location Analysis
http://www.ent.ohiou.edu/~thale/sola/sola.html

Books and Journals

H.A. Eiselt, C.-L. Sandblom.
Decision Analysis, Location
Models, and Scheduling Problems
Springer. 2004.

Z. Drezner, H. Hamaher (Eds.)
Facility Location.
Applications and Theory
Springer. 2004.

Z. Drezner (Ed.)
Facility Location. A Survey
of Applications and Methods
Springer. 1995.

P.B. Mirchandani, R.L. Francis (Eds.)
Discrete Location Theory. Wiley & Sons. 1990

Studies and Locational Analysis.
ISENLORE. Editor-in-Chief F.Plastria

Theoretical results

- The UFL problem is strongly NP-hard even for metric case.
- An 1,463–factor approximation algorithm for the metric UFL problem would imply P = NP (Guna, Khuller 1999; Sviridenko).
- An 1,52–factor approximation algorithm for the metric UFL problem (Mahdian, Ye, and Zhang 2002).
- An ε–factor approximation algorithm for any $\varepsilon > 0$ in the special case when facilities and users are points in the plane and service costs are geometrical distances (Arora, Raghavan, Rao 1998; Kolliopoulos, Rao 1999).
- There is no constant–factor approximation algorithm for general UFL problem if $P \neq NP$.

Empirical Results

- Efficient branch and bound methods based on the fast heuristics for dual problem (Lebedev, Kovalevskaya 1974; Trubin 1973; Beresnev 1974; Bilde, Krarup 1977; Erlenkotter 1978).
- Fast randomized heuristics for large scale metric instances (Chudak, Barahona 2000).
- Improved branch and bound method for large scale instances (Hansen, Mladenovic 2003).
- Effective and efficient local search methods for large scale instances (Resende, Werneck 2002, 2004; Hansen, Mladenović 1997)
- Benchmark library “Discrete Location Problems” (Kochetov, Kochetova, Paschenko, Alexseeva, Ivanenko 2004)

Reduction to Pseudo –Boolean Functions

For a given vector g_i, $i\in I$, with ranking

$$g_{i_1} \le g_{i_2} \le \ldots \le g_{i_m}, \quad m = |I|$$

we introduce a vector Δg_i, $i = 0,\ldots, m$

$$\Delta g_0 = g_{i_1}$$

$$\Delta g_l = g_{i_{l+1}} - g_{i_l}, \quad 1 \le l \le m-1$$

$$\Delta g_m = -g_{i_m}.$$

Lemma. For arbitrary vector $z_i \in \{0,1\}$, $i\in I$, $z \ne (1,\ldots,1)$ we have

$$\min_{i|z_i=0} g_i = \sum_{l=0}^{m-1} \Delta g_l z_{i_1} \ldots z_{i_l}.$$

$$\max_{i|z_i=0} g_i = -\sum_{l=0}^{m-1} \Delta g_{m-l} z_{m-l+1} \ldots z_{i_m}$$

Hence, we can get a pseudo–Boolean function for UFL problem:

$$p(z) = \sum_{i \in I} f_i (1 - z_i) + \sum_{j \in J} \sum_{l=0}^{n-1} \Delta c_{lj} z_{i_1^j} \ldots z_{i_l^j},$$

where the ranking $i_1^j, \ldots, i_m^j$ is generated by column j of the matrix (c_{ij}):

$$c_{i_1^j} \le c_{i_2^j} \le \ldots \le c_{i_m^j}, \quad j \in J$$

and for optimal solutions we have

$$\begin{cases} x_i^* = 1 - z_i^*, \quad i \in I \\ S^* = \{i \in I \mid x_i^* = 1\} \\ F(S^*) = p(z^*) \end{cases}$$

Example

$I = \{1, 2, 3\}, \quad J = \{1, 2, 3\}$ and

$$f_i = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}; \; c_{ij} = \begin{pmatrix} 0 & 3 & 10 \\ 5 & 0 & 0 \\ 10 & 20 & 7 \end{pmatrix}.$$

Pseudo–Boolean function:

$$p(z) = 10(1 - z_1) + 10(1 - z_2) + 10(1 - z_3) + (5z_1 + 5z_1z_2) + (3z_2 + 17z_1z_2) +$$

$$+ (7z_2 + 3\, z_2z_3) = 15 + 5(1 - z_1) + 0(1 - z_2) + 10(1 - z_3) + 22z_1z_2 + 3\, z_2z_3\, .$$

New UFL problem: $I' = I, \; J' = \{1, 2\}$

$$f_i' = \begin{pmatrix} 5 \\ 0 \\ 10 \end{pmatrix}; \; c_{ij}' = \begin{pmatrix} 0 & 3 \\ 0 & 0 \\ 22 & 0 \end{pmatrix}.$$

From PB Function to UFL Problem

PB Function with positive coefficients for nonlinear items:

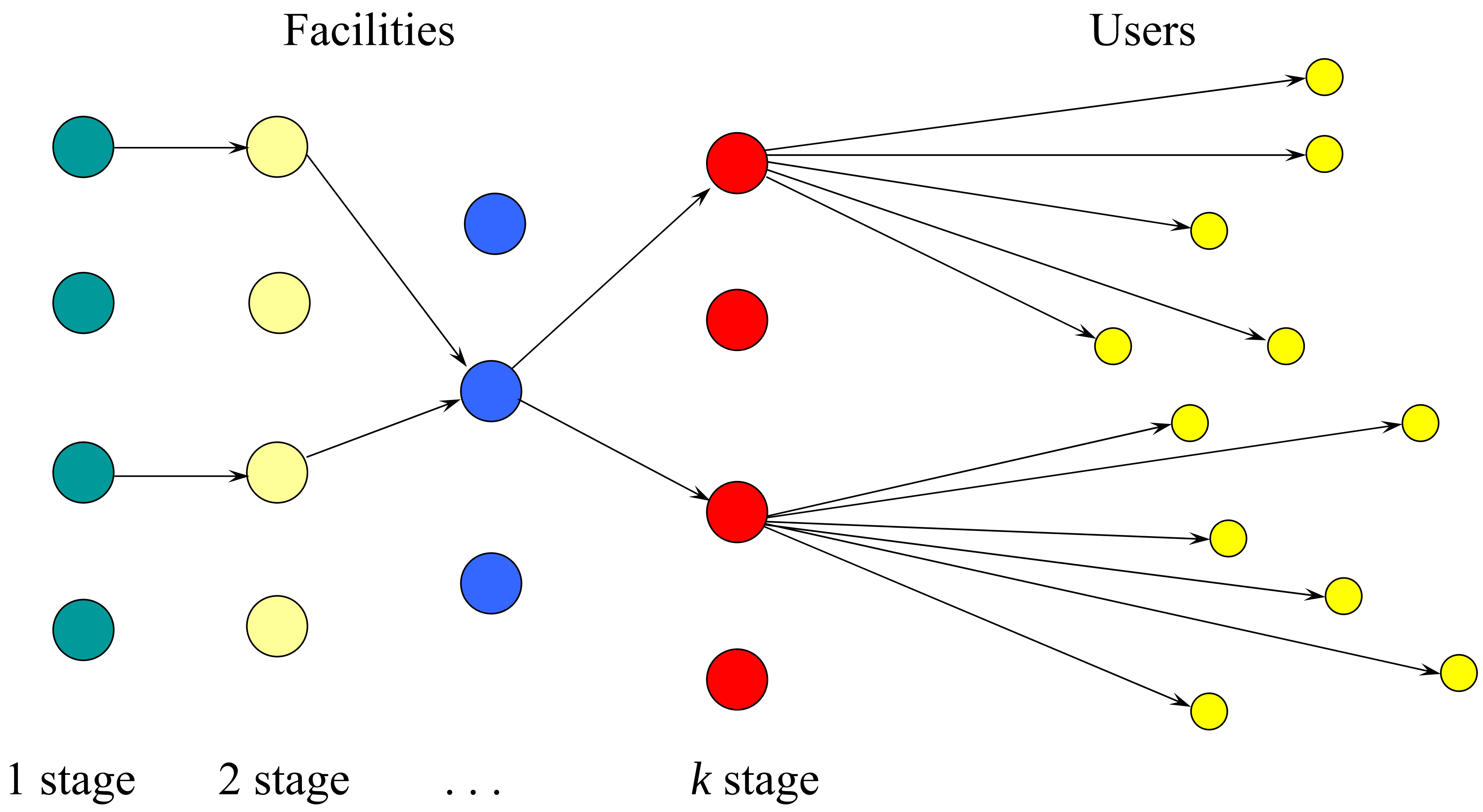
$$p(z) = \sum_{i \in I} f_i (1 - z_i) + \sum_{l \in L} a_l \prod_{i \in I_l} z_i$$

where f_i, $a_l > 0$, $I_l \subset I$, $l \in L$.

Theorem. For given pseudo-Boolean function $p(z)$ an equivalent instance of UFL problem with minimal number of users can be found in polynomial time.

Proof. The family I_l, $l \in L$ with relation $I_{l_1} < I_{l_2} \Leftrightarrow I_{l_1} \subset I_{l_2}$ is partially ordered set (*poset*). Chain in poset is a sequence $I_{l_1} < I_{l_2} \ldots < I_{l_k}$. Each chain generates an element of the set J. We need to partition the family I_l, $l \in L$ into the minimal number of chains. It can be done in polynomial time (see Dilworth Theorem). ■

Multi Stage Facility Location Problem



- Input:
  - a set J of users;
  - a set I of potential facilities;
  - a set P of potential facility paths;
  - a (0,1)-matrix (q_{pi}) of inclusions facilities into paths;
  - a fixed cost f_i of opening facility i;
  - a production-transportation cost c_{pj} to service user j from facility path p;
- Output:

  a set $P' \subseteq P$ of facility paths;
- Goal:

  minimize the total cost to open facilities and service all users

$$\min_{P' \subseteq P} \left\{ \sum_{i \in I} \max_{p \in P'} f_i q_{p_i} + \sum_{j \in J} \min_{p \in P'} c_{pj} \right\}.$$

Integer Programming Formulation

Variables:

$$x_i = \begin{cases} 1 & \text{if facility } i \text{ is open,} \\ 0, & \text{otherwise,} \end{cases}$$

$$y_{pj} = \begin{cases} 1 & \text{if user } j \text{ is serviced by facility path } p, \\ 0, & \text{otherwise.} \end{cases}$$

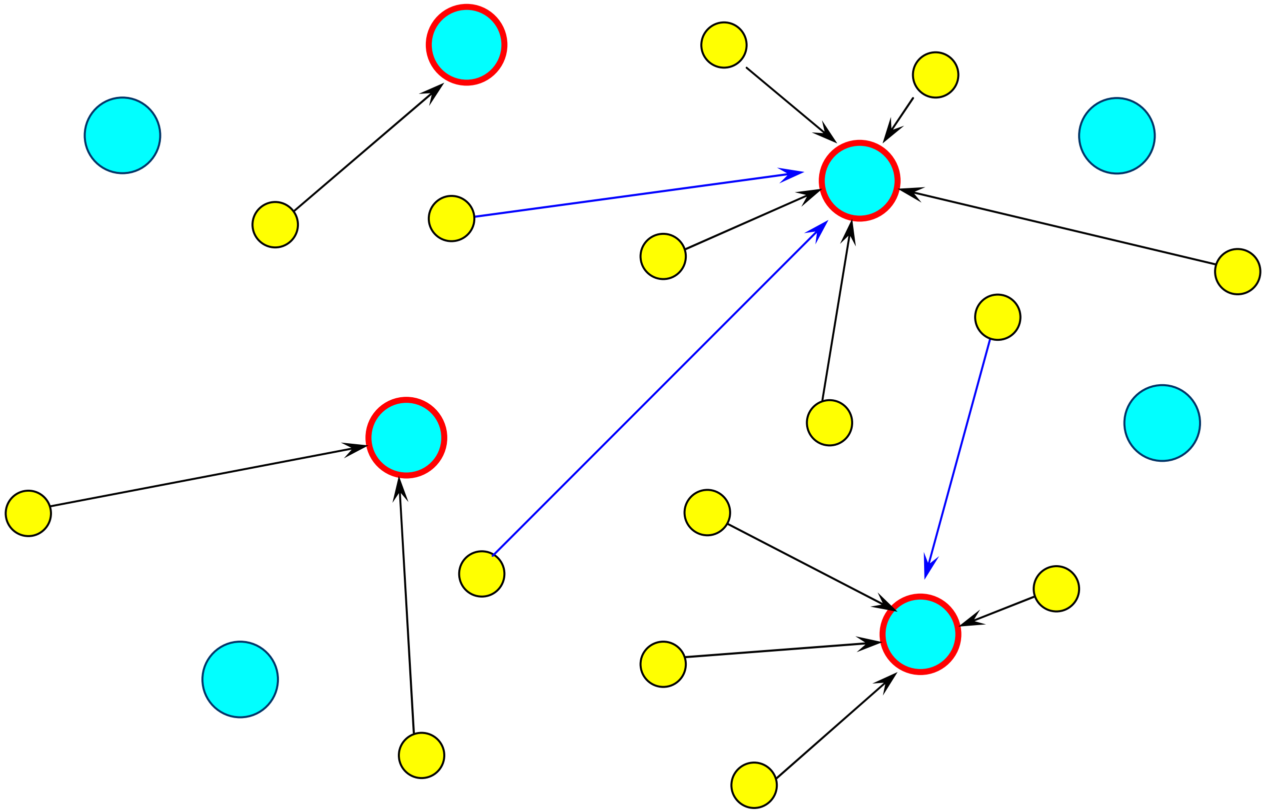
Mathematical model: $$\min \left\{ \sum_{i \in I} f_i x_i + \sum_{p \in P} \sum_{j \in J} c_{pj} y_{pj} \right\}$$

s.t. $$\sum_{p \in P} y_{pj} = 1, \quad j \in J;$$

$$x_i \geq \sum_{p \in P} q_{pi} y_{pj}, \quad j \in J, i \in I;$$

$$x_i, y_{pj} \in \{0,1\}, \quad i \in I, j \in J, p \in P.$$

UFL Problem with User Preferences (UFLPUP)



$I = \{1,\ldots, 8\}$ is potential facility locations;

$J = \{1,\ldots, 15\}$ is set of users

User is serviced by the most desirable facility.

- Input:
  - a set J of users;
  - a set I of potential facilities;
  - a fixed cost f_i of opening facility i;
  - a production-transportation cost c_{ij} to service user j from facility i;
  - a user preferences d_{ij}: facility i_1 is more desirable then i_2 for user j if $d_{i_1 j} < d_{i_2 j}$.
- Output:

  a set $S \subseteq I$ of opening facilities;
- Goal:

  minimize the total cost to open facilities and service all users

$$F(S) = \sum_{i \in S} f_i + \sum_{j \in J} \min_{i \in S} c_{i(s_j)j}, \quad \text{where} \quad i(s_j) = \arg\min_{i \in S} d_{ij}, \quad j \in J.$$

Integer Programming Formulation

Variables:

$$x_i = \begin{cases} 1 & \text{if facility } i \text{ is open,} \\ 0, & \text{otherwise,} \end{cases} \qquad y_{ij} = \begin{cases} 1 & \text{if user } j \text{ is served by facility } i, \\ 0, & \text{otherwise,} \end{cases}$$

Mathematical model:

Company: $$\min_x \left\{ \sum_{i \in I} f_i x_i + \sum_{i \in I} \sum_{j \in J} c_{ij} y_{ij}^*(x) \right\} \quad \text{s.t. } x_i \in \{0,1\} \quad i \in I;$$

where $y_{ij}^*(x)$ is optimal solution of the user problem:

Users: $$\min_y \sum_{j \in J} \sum_{i \in I} d_{ij} y_{ij}$$

s.t. $$\sum_{i \in I} y_{ij} = 1, \quad j \in J;$$

$$y_{ij} \le x_i, \quad x_i, y_{ij} \in \{0,1\} \quad i \in I, j \in J.$$

Reduction to Pseudo –Boolean Functions

Let the ranking for user $j \in J$ be

$$d_{i_1 j} < d_{i_2 j} < \ldots < d_{i_m j}, \quad m = |\, I \,|$$

$$S_{ij} = \{k \in I \mid d_{kj} < d_{ij}\}$$

and $\nabla C_{i_1 j} = C_{i_1 j}, \ldots, \nabla C_{i_l j} = C_{i_l j} - C_{i_{l-1} j}, \;\; 1 < l \le m;$

We get the equivalent minimization problem for Pseudo–Boolean function

$$P(z) = \sum_{i \in I} f_i (1 - z_i) + \sum_{j \in J} \sum_{i \in I} \nabla C_{ij} \prod_{k \in S_{ij}} z_{kj}$$

From PB Function to UFPUP Problem

We are given

$$P(z) = \sum_{l \in L} a_l \prod_{i \in I_l} z_i$$

with arbitrary coefficients a_l, $l \in L$.

Theorem 2. For PB Function $P(z)$ the correspondence UFLP Problem with minimal number of us will minimal cardinality of the set J can be found in polynomial time.

Proof. (similar to previous statement).

Competitive Facility Location Problem

- Input:
  - a set J of users;
  - a set I of potential facilities;
  - a demand d_j of user j;
  - a distance function $c : I \times J \to R$;
  - a number of facilities p_0 to open by Leader;
  - a number of facilities p_1 to open by Follower;
- Output:

  a set $S \subset I$, $|S| = p_0$ of opening facilities by Leader;
- Goal:

  maximize the total number of users which are served by Leader.

The Leader variables:

$$z_i = \begin{cases} 1 & \text{if facility } i \text{ is open by Leader} \\ 0, & \text{otherwise;} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if the demound of user } j \text{ is satisfied by Leader} \\ 0, & \text{otherwise;} \end{cases}$$

The Follower variables:

$$x_i = \begin{cases} 1 & \text{if facility } i \text{ is open by Follower} \\ 0, & \text{otherwise;} \end{cases}$$

$$\bar{y}_j = \begin{cases} 1 & \text{if the demound of user } j \text{ is satisfied by Follower} \\ 0, & \text{otherwise;} \end{cases}$$

The set of appropriate point for Follower

$I_j(z) = \{i \in I \mid c_{ij} < \min\,(c_{kj} \mid z_k = 1)\}$

Mathematical model

Leader:

$$\max_{y,z\in\{0,1\}} \sum_{j\in J} d_j y_j$$

s.t.

$$y_j \le 1 - x_i^*, \quad i \in I_j(z), j \in J;$$

$$\sum_{i\in I} z_i \le p_0,$$

where x_i^* is optimal solution of the problem:

Follower:

$$\max_{\bar{y},x\in\{0,1\}} \sum_{j\in J} d_j \bar{y}_j$$

s.t.

$$\bar{y}_j \le \sum_{i\in I_j(z)} x_i, \quad j \in J;$$

$$\sum_{i\in I} x_i \le p_1;$$

$$z_i + x_i \le 1, \quad i\in I.$$

Mathematical model for K Followers

Leader:

$$\max_{y,z\in\{0,1\}} \sum_{j\in J} d_j y_j;$$

s.t.

$$y_j \le 1 - x^*_{ik}, \quad i \in I_j(z), j \in J, k \in K;$$

$$\sum_{i\in I} z_i \le p_0,$$

where x^*_{ik} is optimal solution of the problem:

Follower k:

$$\max_{\bar{y},x\in\{0,1\}} \sum_{j\in J} d_j \bar{y}_{jk};$$

s.t.

$$\bar{y}_{jk} \le \sum_{i\in I_{jk-1}(z,x)} x_{ik}, \quad j \in J;$$

$$\sum_{i\in I} x_{ik} \le p_k;$$

$$z_i + \sum_{k'=1}^{k-1} x^*_{ik'} + x_{ik} \le 1, \quad i \in I.$$

The set of appropriate point for Follower k: $I_{jk}(z,x) = \{i \in I \mid c_{ij} < \min(c_{lj} \mid z_l + \sum_{k'=1}^{k} x^*_{lk'} > 0)\}$