Facility location problems

Discrete models and local search methods

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Lecture 5 Computationally Difficult Instances for the Uncapacitated Facility Location Problems

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- Instances based on Binary Perfect Codes
- Instances based on the torus which can be obtained from a chessboard by identification of the boundaries
- Instances based on Finite Projective Planes
- Instances with large duality gap
- Instances with clustering local optima into several galaxies

The Uncapacitated Facility Location Problem

- Input:
  - a set J of users;
  - a set I of potential facilities;
  - a fixed cost f_i to open facility i;
  - a production-transportation cost c_{ij} to service user j from facility i.
- Output:

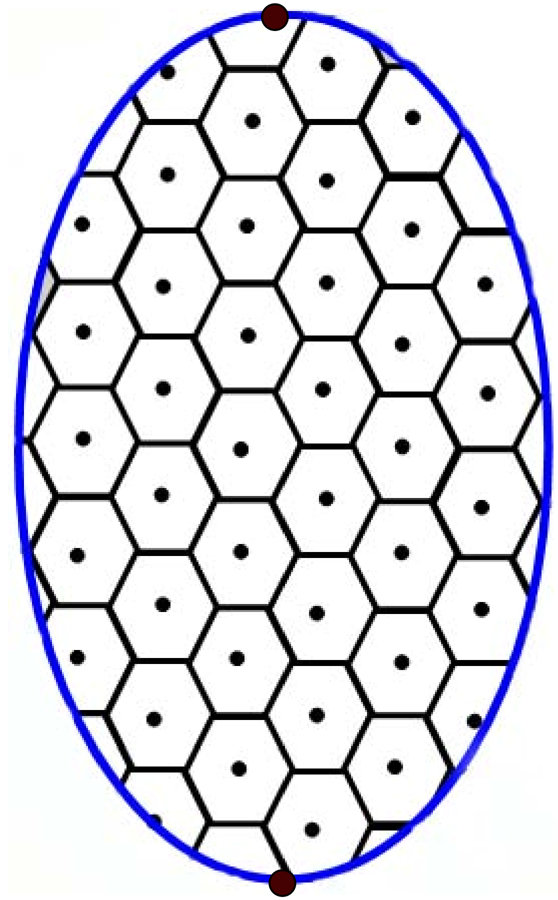
  a set $S \subseteq I$ of opened facilities;
- Goal:

  minimize the total cost to open facilities and service all users

$$F(S) = \sum_{i \in S} f_i + \sum_{j \in J} \min_{i \in S} c_{ij} .$$

Instances on perfect codes

Binary perfect codes with distance 3 is a subset $C \subset B_k$,
$|C| = 2^k/(k+1)$ such that $d(c_1,c_2) \geq 3$ for all $c_1,c_2 \in C$, $c_1 \neq c_2$
Each perfect code produces a partition of the hypercube into $2^k/(k+1)$ disjoint spheres of radius 1.
$N(C)$ is a number of perfect codes.



Theorem 5.1. [Krotov]

$$N(C) \geq 2^{2^{\frac{k+1}{2} - \log_2(k+1)}} \cdot 3^{2^{\frac{k-3}{4}}} \cdot 2^{2^{\frac{k+5}{4} - \log_2(k+1)}}.$$

Theorem 5.2. [Solov′eva]

Minimal distance between codes is $2^{(k+1)/2}$.

http://www.codingtheory.gorodok.net/literature/lecture-notes.pdf

Random instances on perfect codes

Each perfect code produces a partition of the hypercube and corresponds to a strong local optimum under ($Flip \cup Swap$)–neighborhood.

Define $I = J = \{1, \ldots, 2^k\}$ and

$$c_{ij} = \begin{cases} \xi_{ij} & \text{if } d(x_i, x_j) \le 1 \\ +\infty, & \text{otherwise} \end{cases}, \qquad \xi_{ij} \text{ is a random number}$$

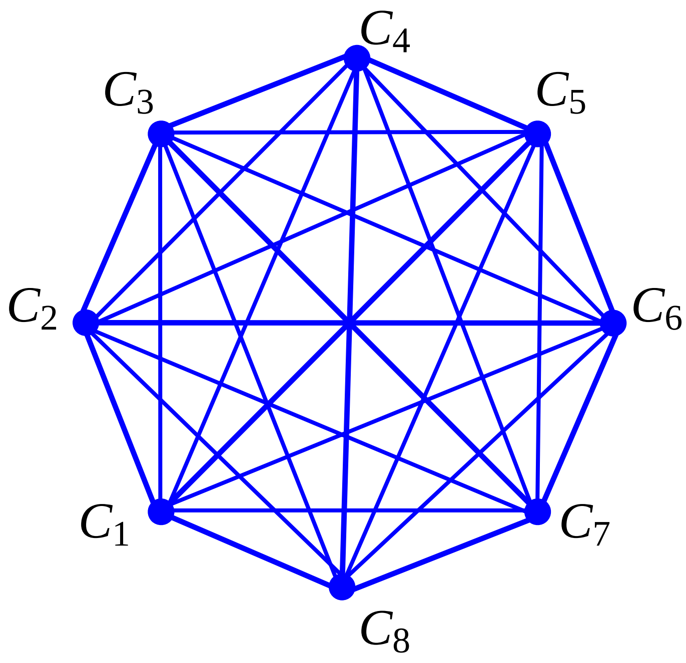
$$f_i = f > \sum_{i \in I} \sum_{j \in J} \xi_{ij}, \; i \in I.$$

For $k = 7$ we have $N(C) = 280 = 35 \times 8$.

For each 8 codes, the pair distance is 32.

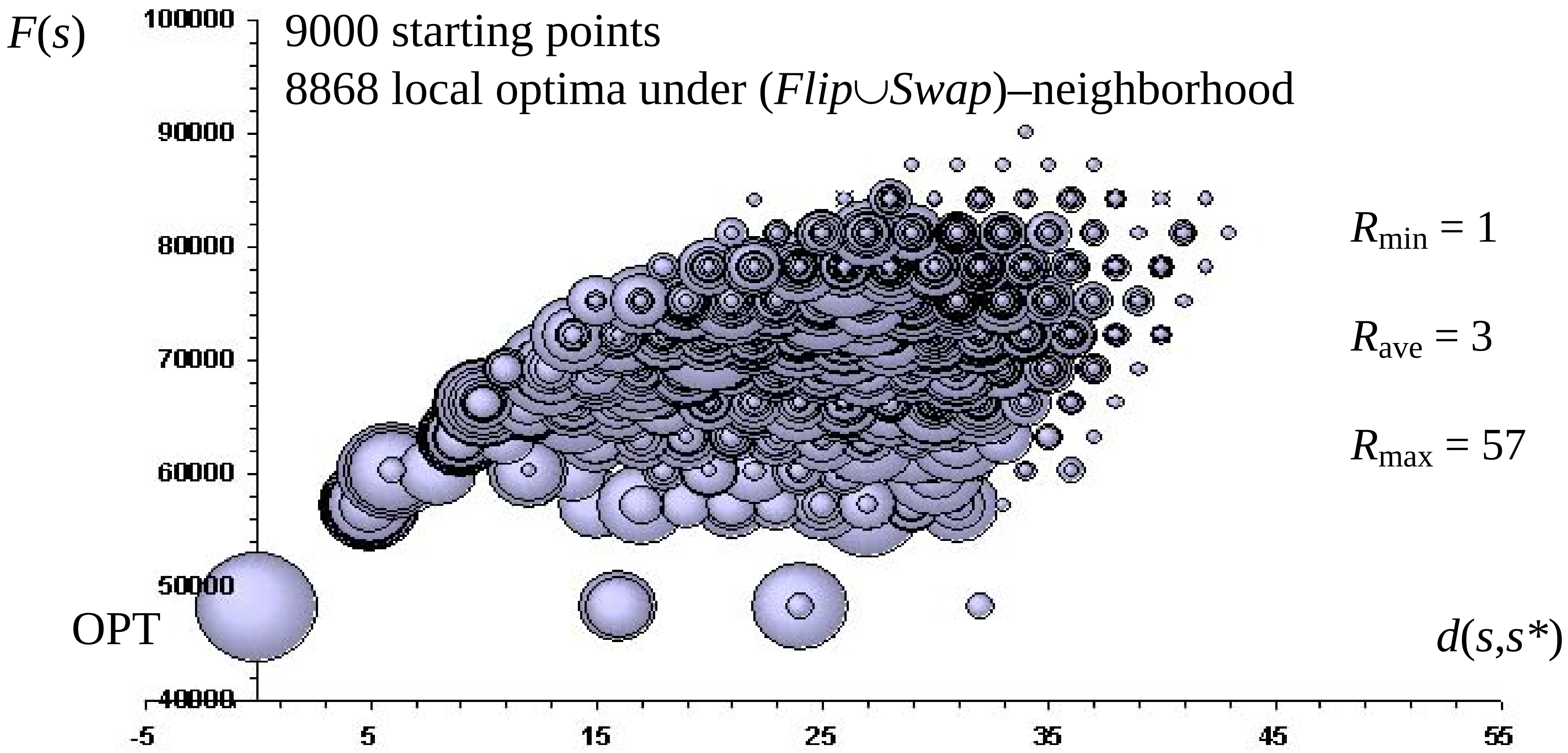
Maximal distance between codes is 32.

Minimal distance between codes is 16.



$d(C_i, C_j) = 32$

The costs of local optima against their distances from global optimum



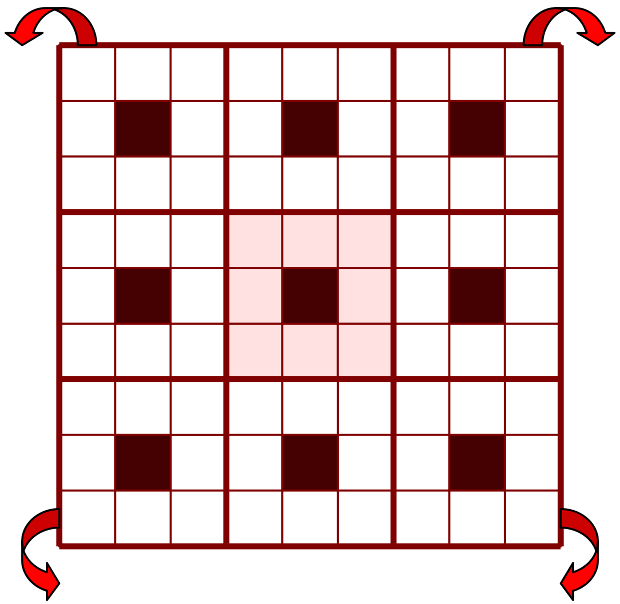
The radius R of each sphere is a number of local optima which are located near the given local optimum, $d(s, s') \leq 10$. $R(\text{OPT}) = 52$, $D = 55$.

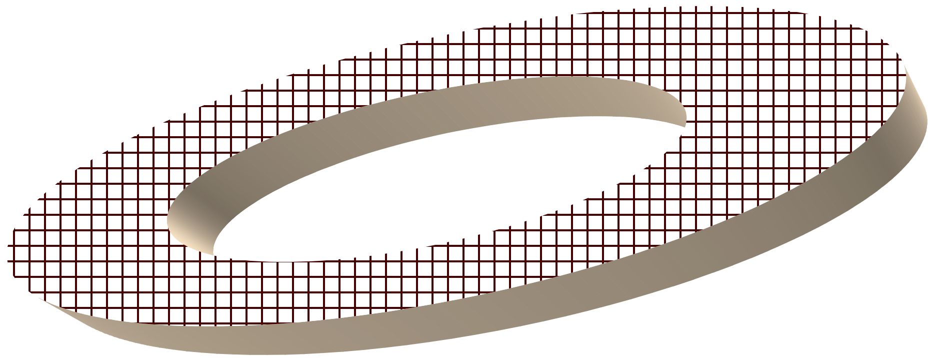
Instances on Chess-Board

Let us glue the boundaries of the $3k \times 3k$ chess board so that to get a torus.

Put $r = 3k$. Each cell of torus has 8 neighboring cells.

For example, the cell (1, 1) has the following neighbors: (1, 2), (1, r), (2, 1), (2, 2), (2, r), (r, 1), (r, r). The torus is divided into k^2 squares by 9 cells in each of them.





Random instances on Chess-Board

Define $n = 9k^2$, $I = J = \{1,\ldots, n\}$ and

$$c_{ij} = \begin{cases} \xi_{ij} & \text{if the cells } i, j \text{ are neighbors} \\ +\infty, & \text{otherwise} \end{cases}, \quad \xi_{ij} \text{ is a random number}$$

$$f_i = f > \sum_{i \in I} \sum_{j \in J} \xi_{ij}, \quad i \in I.$$

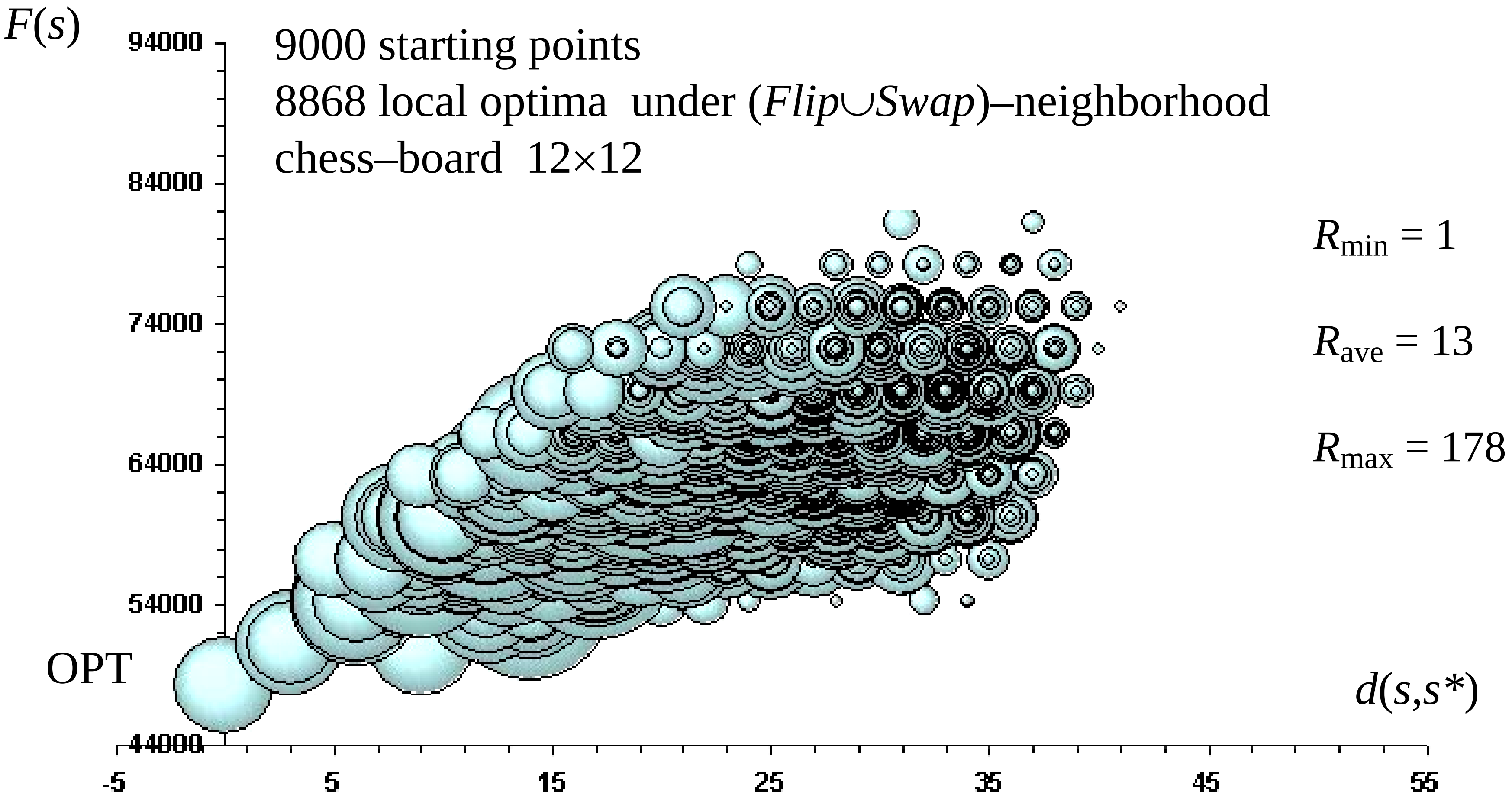
The torus is divided into k^2 squares. Every cover of the torus corresponds to a strong local optimum for the UFL problem with ($Flip \cup Swap$)–neighborhood.

The total number of partitions is $2 \cdot 3^{k+1} - 9$.

The minimal distance between them is $2k$.

The maximal distance between them is k^2.

The costs of local optima against their distances from global optimum



The radius R of each sphere is a number of local optima which are located near the given local optimum, $d(s, s') \leq 10$. $R(\text{OPT}) = 53$, $D = 50$.

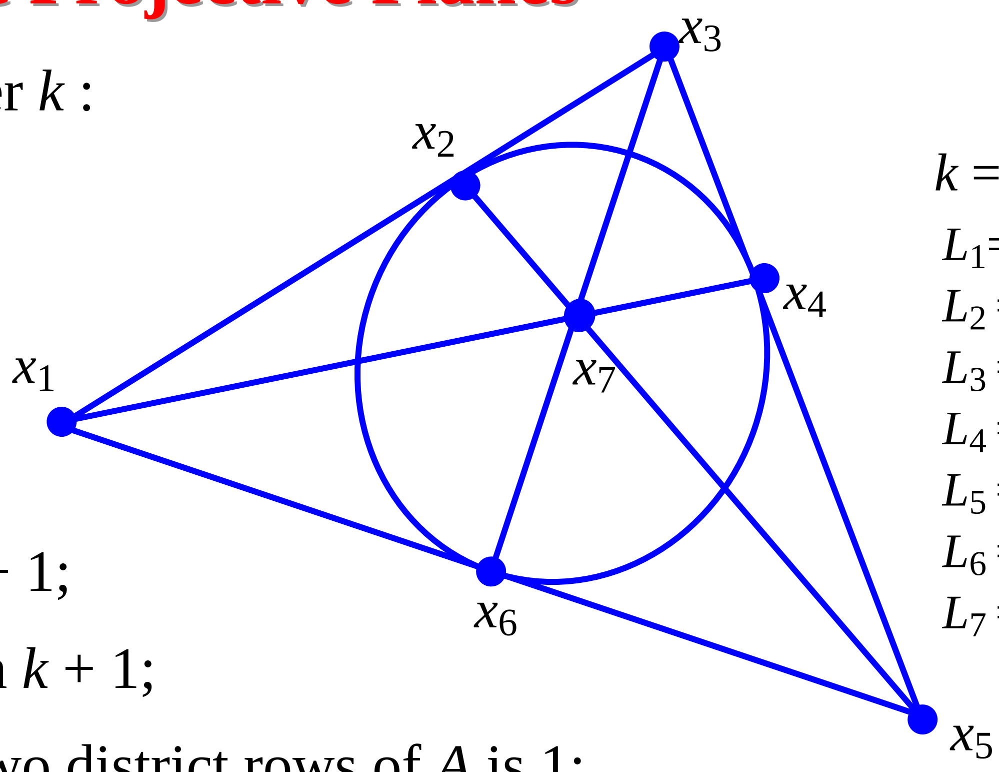
Instances on the Finite Projective Planes

Finite projective plane of order k :

$n = k_2 + k + 1.$

Points $x_1,\dots x_n$.

Lines $L_1,\dots, L_n$.



$k = 2;\ n = 7$

$L_1 = \{x_1, x_2, x_3\}$
$L_2 = \{ x_3, x_4, x_5\}$
$L_3 = \{ x_1, x_5, x_6\}$
$L_4 = \{ x_1, x_4, x_7\}$
$L_5 = \{ x_3, x_7, x_6\}$
$L_6 = \{ x_2, x_7, x_5\}$
$L_7 = \{ x_2, x_4, x_6\}$

Incidence ($n\times n$) matrix A:

1. A has constant row sum $k + 1$;
2. A has constant column sum $k + 1$;
3. The inner product of any two district rows of A is 1;
4. The inner product of any two district columns of A is 1.

These matrices exist if k is a power of prime.

Bundle for the point x_j : $B_j = \{L_i \mid x_j \in L_i\}$.

Random instances on the Finite Projective Planes

Define $I = J = \{1,\ldots, k_2 + k + 1\}$ and

$$c_{ij} = \begin{cases} \xi_{ij} & \text{if line } L_i \text{ contains point } x_j \\ +\infty, & \text{otherwise} \end{cases}, \quad \xi_{ij} \text{ is a random number,}$$

$$f_i = f > \sum_{i\in I} \sum_{j\in J} \xi_{ij}, \quad i\in I.$$

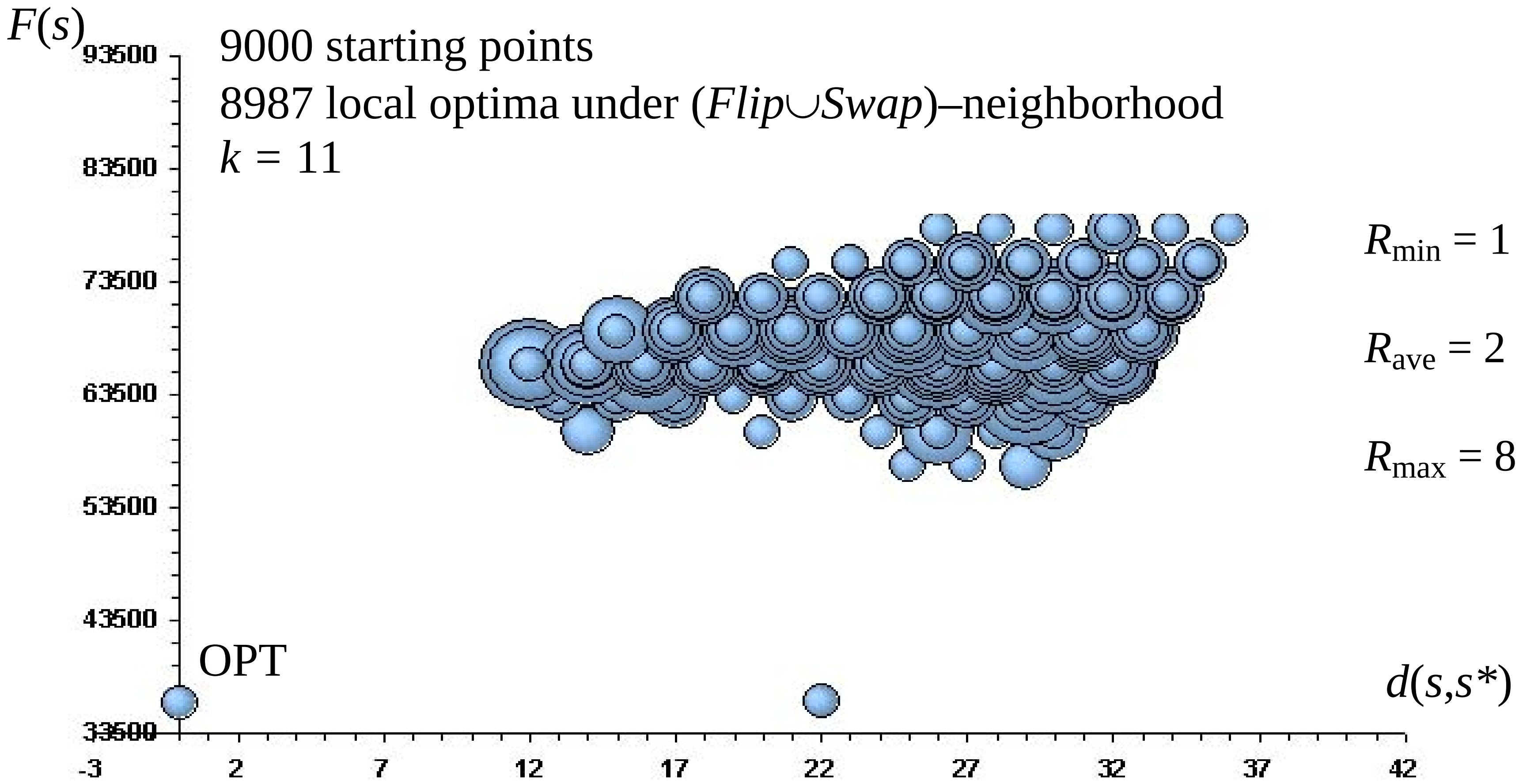
Every bundle corresponds to a strong local optimum of the UFL problem with (*Flip*∪*Swap*)–neighborhood.

Optimal solution corresponds to a bundle of the plane and can be found in polynomial time.

Hamming distance for arbitrary pair of the bundles equals 2*k*.

There are no other local optima with distance less or equal *k* to the bundle.

The costs of local optima against their distances from global optimum



The radius R of each sphere is a number of local optima which are located near the given local optimum, $d(s, s') \le 10$. $R(\text{OPT}) = 1$, $D = 51$.

Random instances with large duality gap

Define $I = J = \{1,\ldots, n\}$ and $f_i = f > \sum_{i\in I} \sum_{j\in J} \xi_{ij}$, $i\in I$.

Gap-A: each column of the matrix (c_{ij}) has l small elements ξ_{ij}.

Gap-B: each row of the matrix (c_{ij}) has l small elements ξ_{ij}.

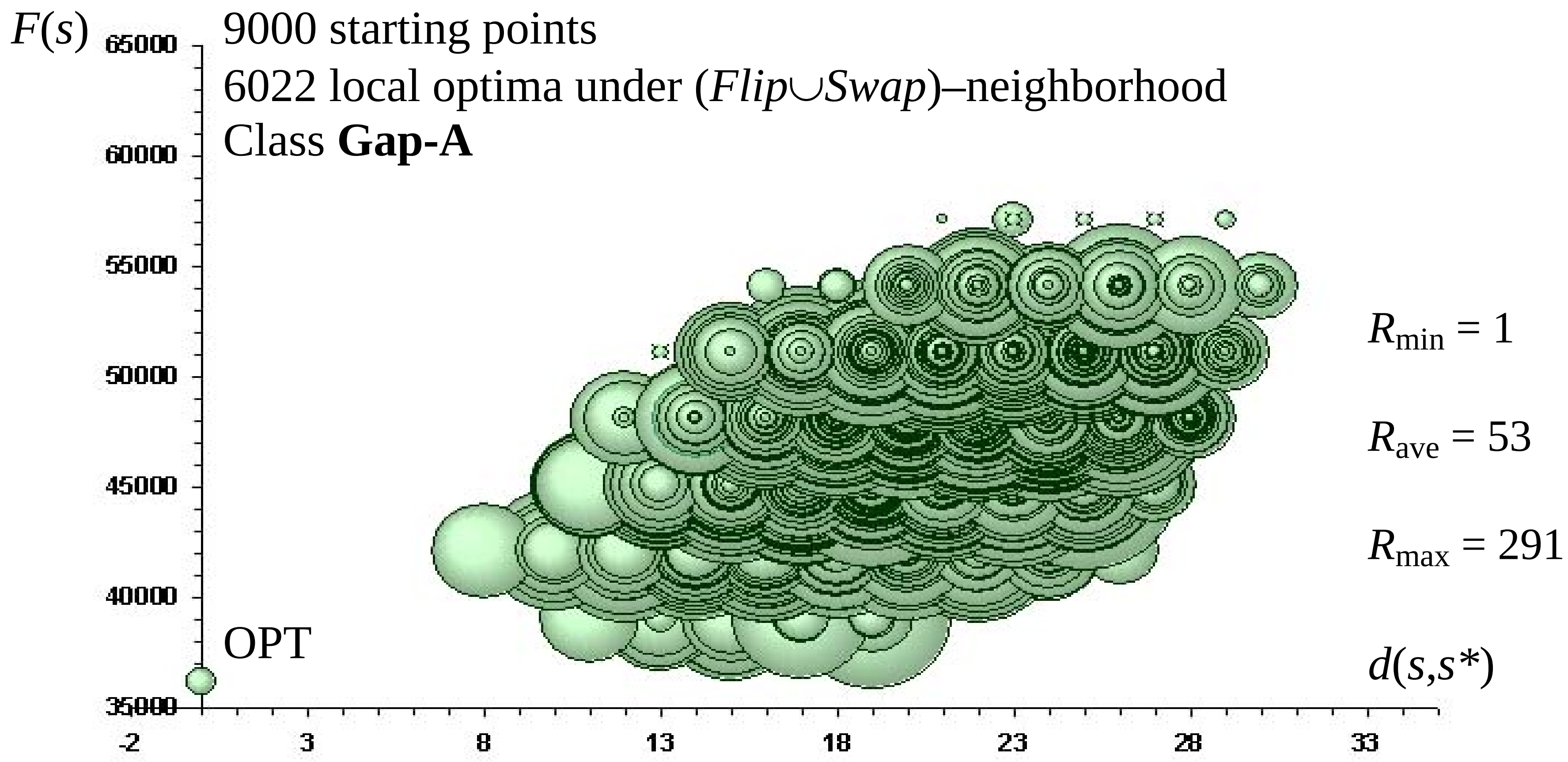
Gap-C: intersection of Gap-A and Gap-B.

The instances have significant duality gap: $\delta = \dfrac{OPT - F_{LP}}{OPT} \cdot 100\%$.

For $l = 10$, $n = 100$ we observe that $\delta\in[21\%, 29\%]$.

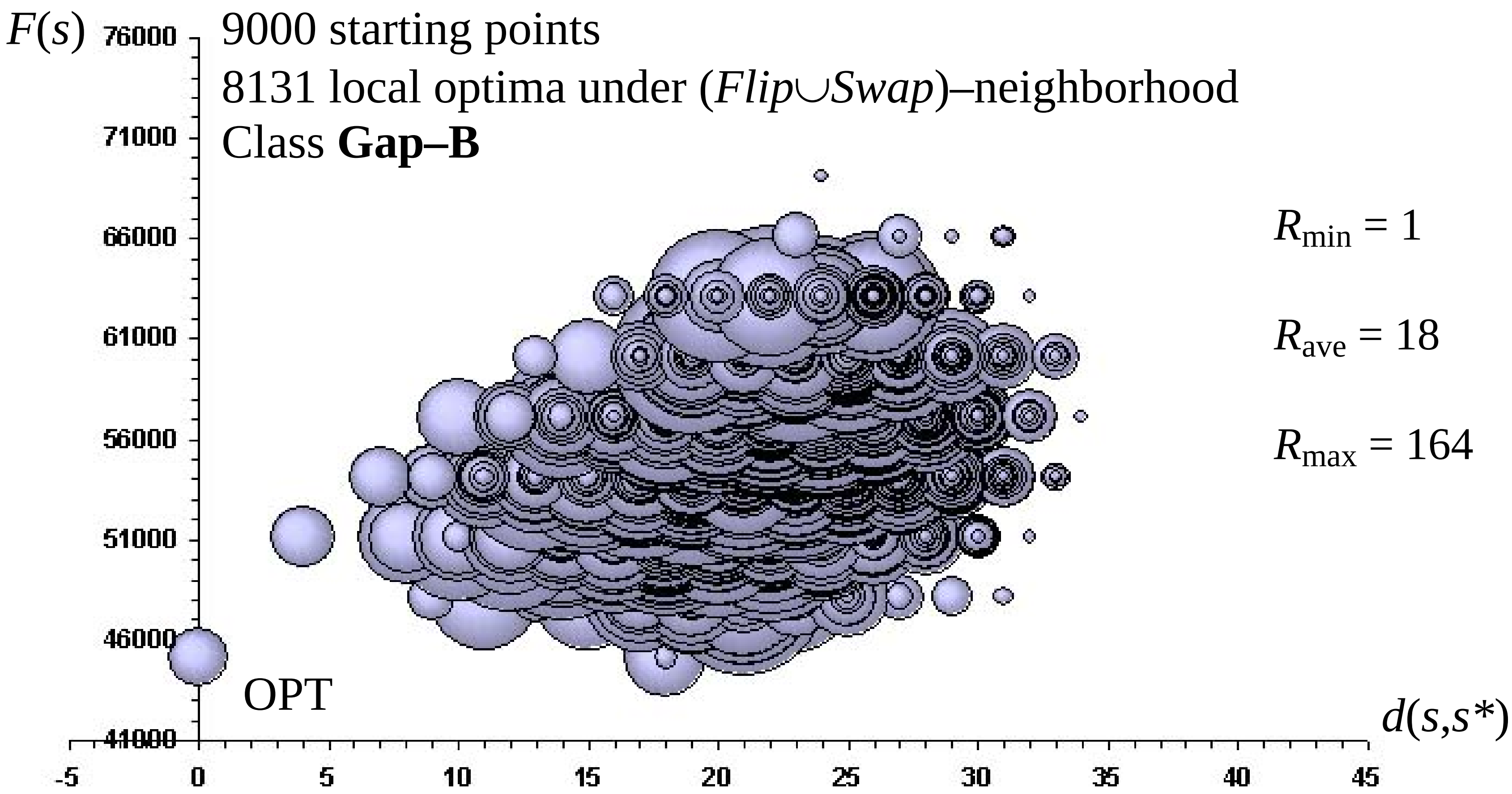
For the class Gap–C, the branch and bound algorithm evaluates about $0{,}5\cdot 10^9$ nodes in the branching tree.

The costs of local optima against their distances from global optimum



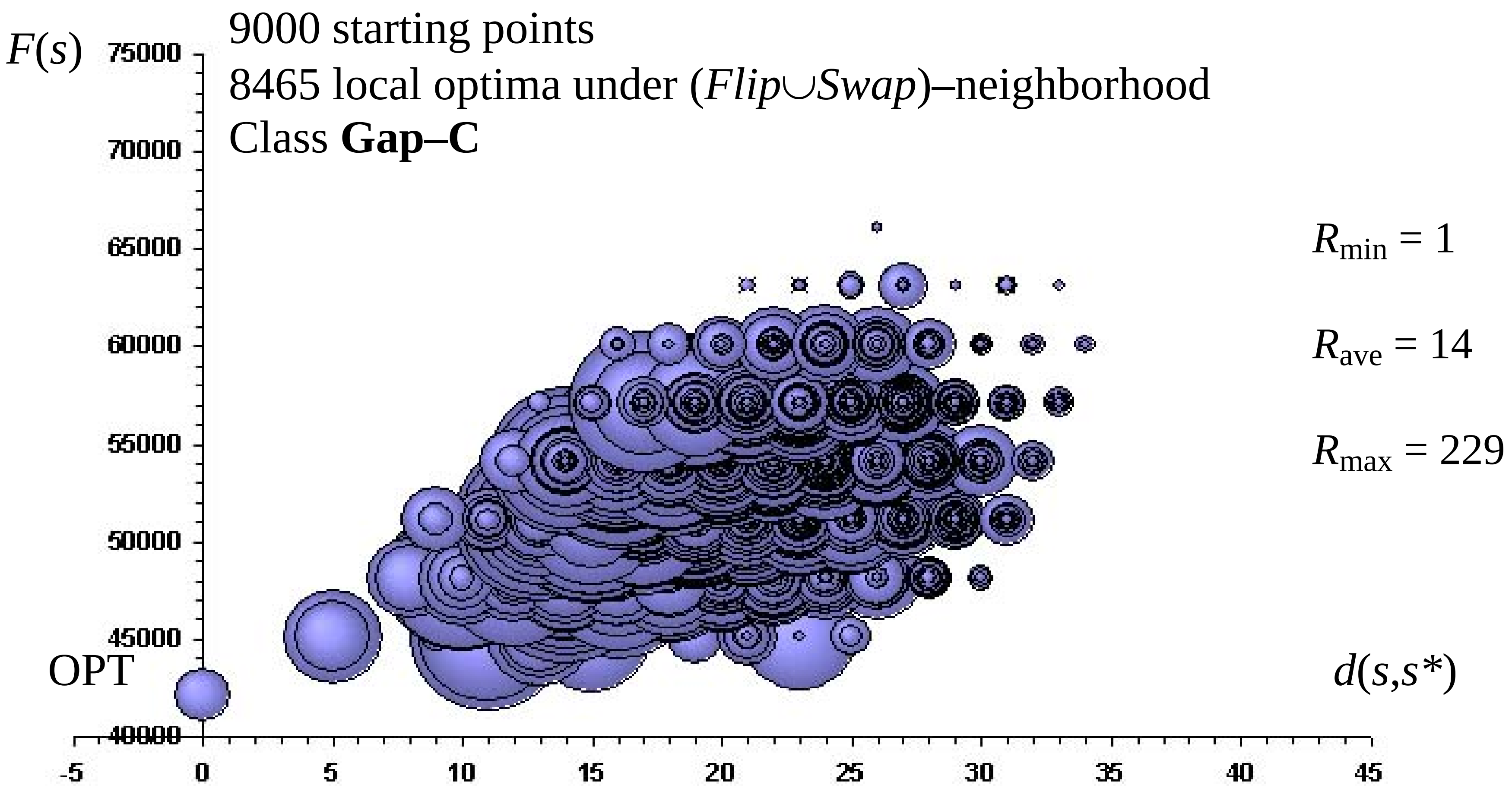
The radius R of each sphere is a number of local optima which are located near the given local optimum, $d(s, s') \leq 10$. $R(\text{OPT}) = 7$, $D = 36$.

The costs of local optima against their distances from global optimum



The radius R of each sphere is a number of local optima which are located near the given local optimum, $d(s, s') \leq 10$. $R(\text{OPT}) = 16$, $D = 42$.

The costs of local optima against their distances from global optimum



The radius R of each sphere is a number of local optima which are located near the given local optimum, $d(s, s') \leq 10$. $R(\text{OPT}) = 21$, $D = 41$.

Easy random instances

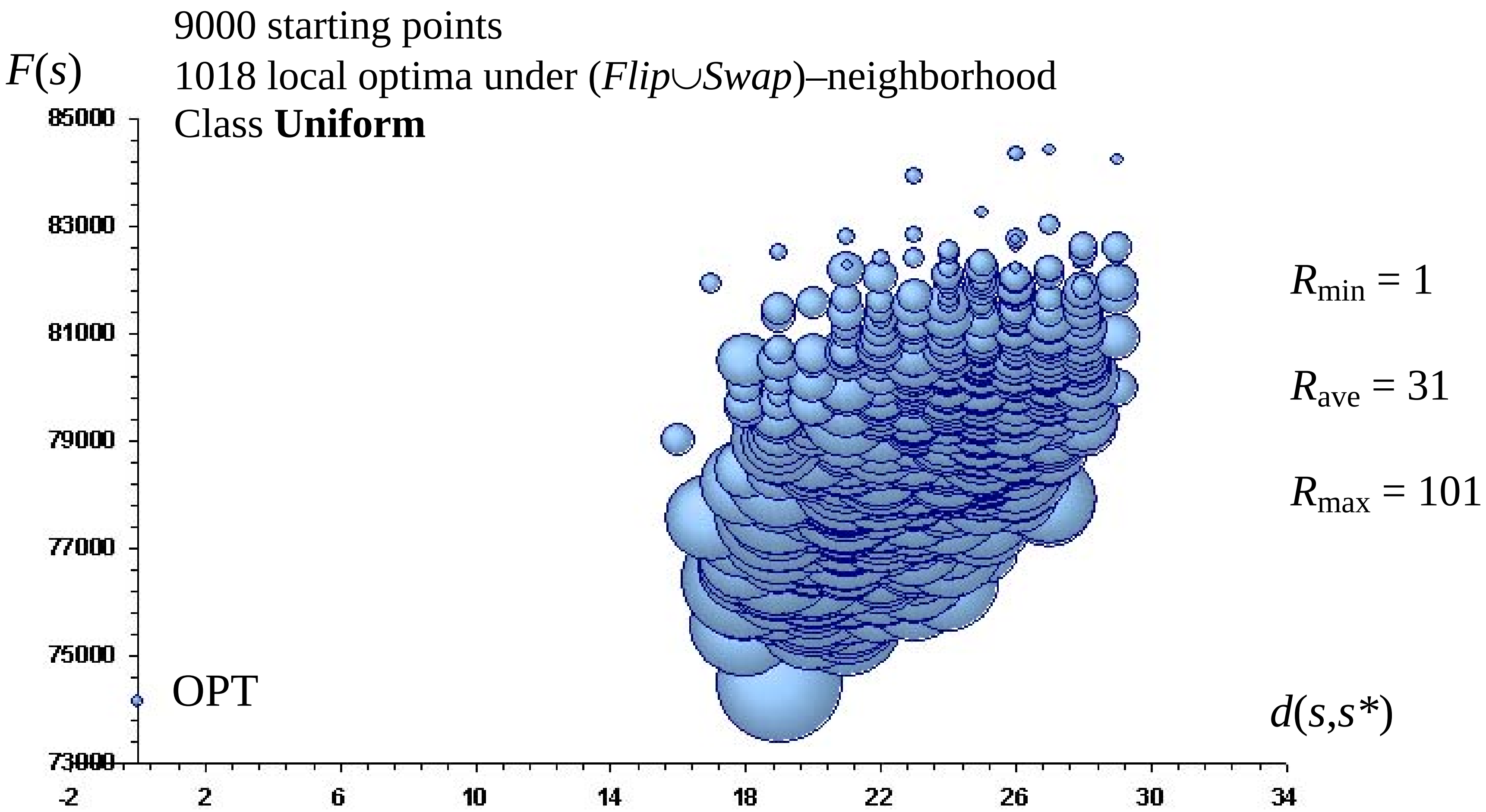
Define $I = J = \{1,\ldots, n\}$ and $f_i = f = 3000$, $i \in I$.

Uniform: values c_{ij} are selected in interval $[0, 10^4]$ at random with uniform distribution and independently from each other.

Euclidean: value c_{ij} is Euclidean distances between points i and j in the two dimension space. The points are selected in square 7000×7000 at random with uniform distribution and independently from each other.

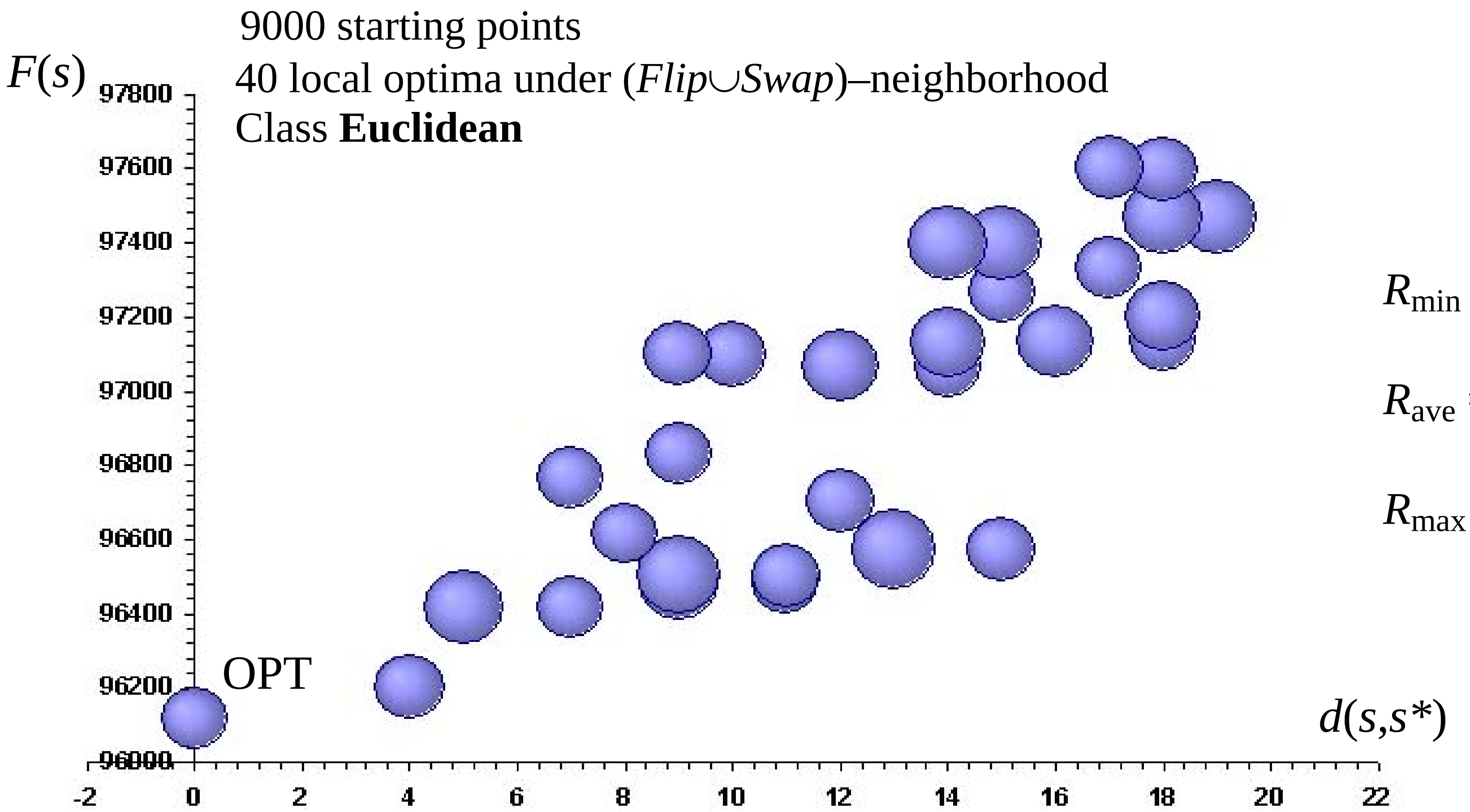
The interval and size of the square are taken in such a way that optimal solutions have the same cardinality as in the previous classes.

The costs of local optima against their distances from global optimum



The radius R of each sphere is a number of local optima which are located near the given local optimum, $d(s, s') \le 10$. $R(\text{OPT}) = 1$, $D = 33$.

Instances on Euclidean plane



R_{min} = 11

R_{ave} = 13

R_{max} = 18

The radius R of each sphere is a number of local optima which are located near the given local optimum, $d(s, s') \leq 10$. R(OPT) = 10, D = 21.

Performance of the B & B algorithm in average

Benchmarks classes	n	Gap δ	Iterations B & B	The best iteration	Running time
BPC_7	128	0.1	374 264	371 646	00:00:15
CB_4	144	0.1	138 674	136 236	00:00:06
FPP_{11}	133	7.5	6 656 713	6 652 295	00:05:20
Gap-A	100	25.6	10 105 775	3 280 342	00:04:52
Gap-B	100	21.2	30 202 621	14 656 960	00:12:24
Gap-C	100	28.4	541 320 830	323 594 521	01:42:51
Uniform	100	4.7	9 834	2 748	00:00:00
Euclidean	100	0.1	1 084	552	00:00:00

Frequency of obtaining optimal solutions by metaheuristics

Benchmarks classes	*n*	PTS	GA	GRASP + LD
BPC_7	128	0.93	0.90	0.99
CB_4	144	0.99	0.88	0.68
FPP_{11}	133	0.67	0.46	0.99
Gap-A	100	0.85	0.76	0.87
Gap-B	100	0.59	0.44	0.49
Gap-C	100	0.53	0.32	0.42
Uniform	100	1.0	1.0	1.0
Euclidean	100	1.0	1.0	1.0

Attributes of the local optima allocation

Benchmarks classes	n	Diameter	Radius			R_{100}	R*
			min	average	max		
BPC_7	8868	55	1	3	57	24	52
CB_4	8009	50	1	13	178	78	53
FPP_{11}	8987	51	1	2	8	3	1
Gap-A	6022	36	1	53	291	199	7
Gap-B	8131	42	1	18	164	98	16
Gap-C	8465	41	1	14	229	134	21
Uniform	1018	33	1	31	101	61	1
Euclidean	40	21	11	13	18	1.	10

Multi Stage Facility Location Problem

- Input:

– a set J of users;

– a set I of potential facilities;

– a set P of potential facility paths;

– a (0,1)-matrix (q_{pi}) of inclusions facilities into paths;

– a fixed cost f_i to open facility i;

– a production-transportation cost c_{pj} to service user j from facility path p;

- Output:

a set $S \subseteq P$ of facility paths;

- Goal:

minimize the total cost to open facilities and service all users

$$F(S) = \sum_{i \in I} \max_{p \in S} f_i q_{p_i} + \sum_{j \in J} \min_{p \in S} c_{pj} .$$

Random instances with clustering local optima

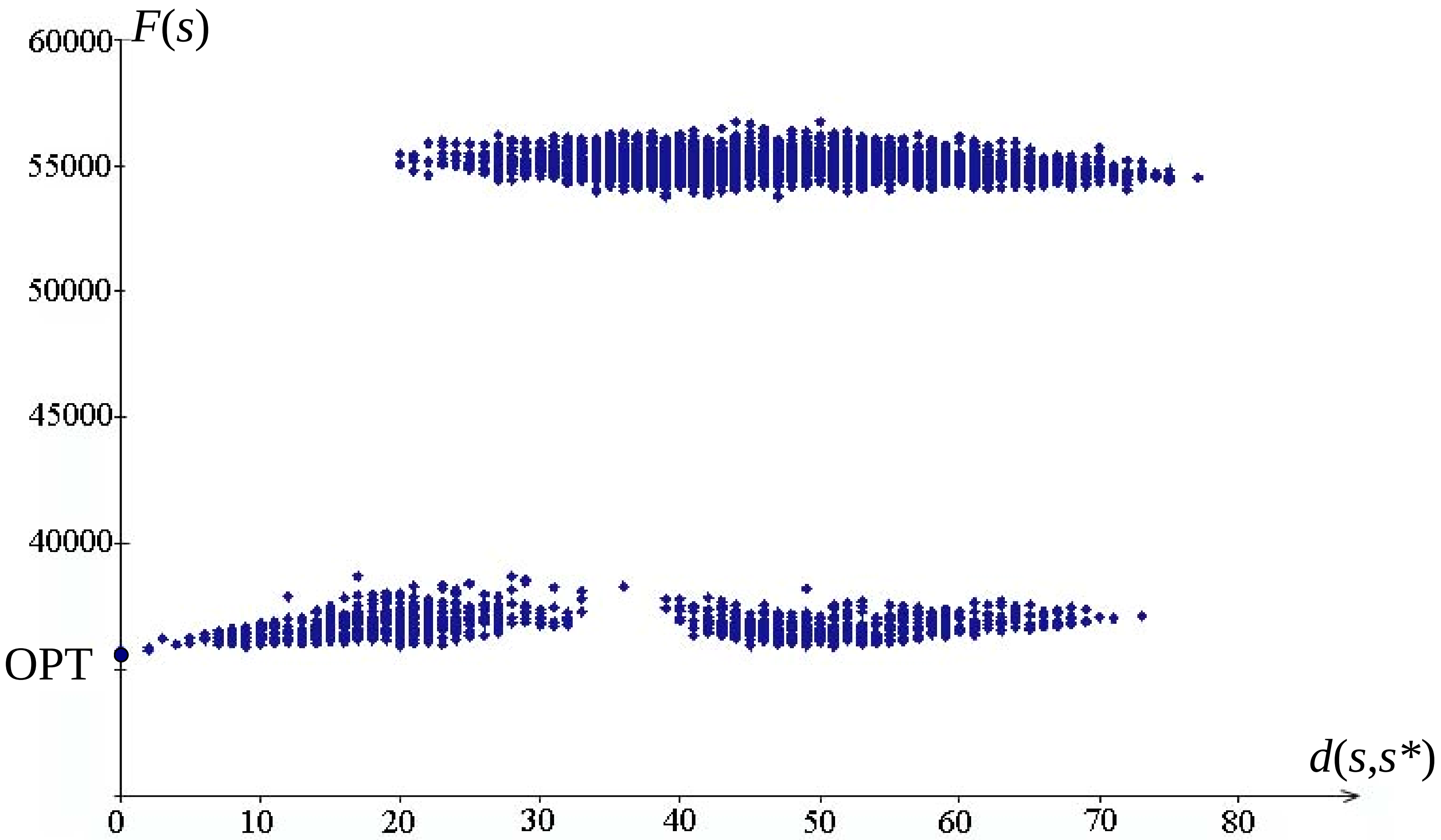
Define $P = J = \{1,\dots, n\}$, $I = \{1,\dots, m\}$ and

$$f_i = \begin{cases} f & \text{if} \quad i \ne i_1 \vee i_2 \\ M_1 & \text{if} \quad i = i_1 \\ M_2 & \text{if} \quad i = i_2 \end{cases} \qquad i \in I,\ M_2 > M_1 >> f > 0.$$

$$q_{p_i} = \left[\begin{array}{c} \\ \\ q'_{p_{i,i \ne i_1 \vee i_2}} \\ \\ \\ \end{array} \right] \begin{bmatrix} 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{bmatrix}$$

Values c_{pj} are Euclidean distances between points $j_1, \dots, j_n$ in the two dimension space. The points are selected at random with uniform distribution and independently from each other.

The costs of local optima against their distances from global optimum



Three galaxies of local optima


Easy case


Easy case

Hard case


Extremely hard case

General information • **Authors** • **Other libraries** • **Russian page**

Discrete Location Problems

Benchmark library

- **Simple Plant Location Problem**
- **Capacitated Facility Location Problem**
- **Multi Stage Uncapacitated Facility Location Problem**
- **P-median Problem**
- **Bilevel Location Problem**

Benchmark library was created thanks to financial support
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http://math.nsc.ru/AP/benchmarks/english.html