Lecture 10. The Bin Packing Problem

The one dimensional bin packing problem is defined as follows. Given a set $L = \{1, \dots, n\}$ of items and theirs weights $w_i \in (0,1), i \in L$. We wish to partition the set L into minimal number m of subsets $B_1, B_2, \dots, B_m$ in such a way that

$$\sum_{i \epsilon B_j} w_i \leq 1, \;\; 1 \leq j \leq m.$$

The sets B_j we will call ***bins***.

In other words, we wish to pack all items in a minimal number of bins.

It is NP-hard problem in the strong sense.

Mathematical Model

Decision variables:

$$y_j = \begin{cases} 1 & \text{if bin } j \text{ is used} \\ 0 & \text{otherwise} \end{cases}; \qquad x_{ij} = \begin{cases} 1 & \text{if item } i \text{ is in bin } j \\ 0 & \text{otherwise} \end{cases}$$

$$\min \sum_{j=1}^{n} y_j$$

s.t.

$$\sum_{i=1}^{n} w_i x_{ij} \le y_j, \quad j = 1, \dots, n;$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \dots, n;$$

$$y_i, x_{ij} \in \{0,1\}, \quad i, j = 1, \dots, n.$$

Can we find the optimal solution for the linear programming relaxation in polynomial time?

Bad News

- There is much symmetry in the model.
- The problem is hard to approximate.

Theorem 1. The existence of a polynomial time $\left(\frac{3}{2}-\varepsilon\right)$-approximation algorithm for any positive ε implies P = NP.

Proof. Let us consider the following NP-complete problem. Given n positive numbers $a_1, \dots, a_n$. Is it possible to partition this set into two subsets A_1, A_2 in such a way that $\sum_{i \in A_1} a_i = \sum_{i \in A_2} a_i$?

We put $C = \frac{1}{2}\sum_{i=1}^{n} a_i$, $w_i = \frac{a_i}{c}$, $i = 1, \dots, n$, and apply our $\left(\frac{3}{2}-\varepsilon\right)$-approximation algorithm. If we get 2 bins, then answer is Yes, otherwise No. It is ***exact*** answer! ■

Strong Heuristic (FFD)

Rank the items by the weights:

$$w_1 \geq w_2 \geq \ \dots \geq w_n$$

and apply the First Fit strategy:

- put the first item in the first bin;
- at the step k, we try to put item k into the used bins and, if it is not possible, we put item k into a new bin.

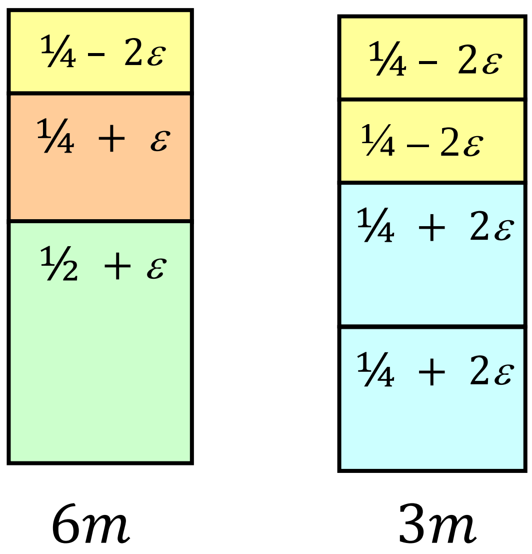
Theorem 2. $FFD(L) \leq \frac{11}{9}\,OPT(L) + 4$ for all L and there exist some instances for the bin packing problem with

$$FFD(L) \geq \frac{11}{9}\,OPT(L).$$

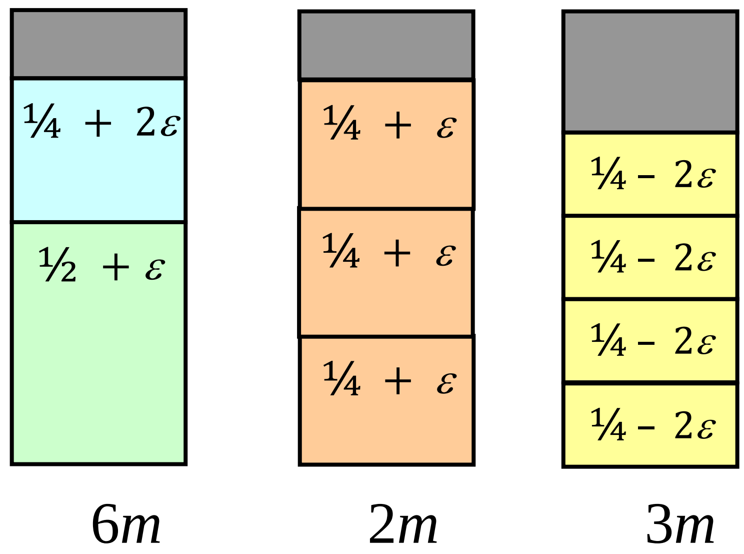
Hard Example

$$L = \{1, \ldots, 30\,m\}$$

$$w_i = \begin{cases} \frac{1}{2} + \varepsilon, & 1 \le i \le 6m \\ \frac{1}{4} + 2\varepsilon, & 6m < i \le 12m \\ \frac{1}{4} + \varepsilon, & 12m < i \le 18m \\ \frac{1}{4} - 2\varepsilon, & 18m < i \le 30m \end{cases}$$



$OPT(L) = 9m$



$FFD(L) = 11m$

Huge Reformulation

Given $L = \{1, \dots, n\}$ is the set of items;

$w_i > 0$ is the weight of item i;

$n_i > 0$, integer, is the number of identical items i

a_{ij} is the number of identical items i in packing pattern j.

Find a partition of all items into a minimal number of bins.

Variables: $x_j \geq 0$, integer, is the number of bins for the pattern j

$$\min \sum_{j \in J} x_j$$

s.t.

$$\sum_{j \in J} a_{ij} x_j \geq n_i, \quad i \in L;$$

$$x_j \geq 0, \text{ integer}, \quad j \in J.$$

J is the set of all possible patterns.

LP-Based Heuristic

Solve the linear programming relaxation

$$\min \sum_{j \in J} x_j$$

s.t.

$$\sum_{j \in J} a_{ij} x_j \geq n_i, \quad i \in L;$$

$$x_j \geq 0, \quad j \in J.$$

Put $x_j = \lceil x_j^* \rceil$, $j \in J$. It is a feasible solution with deviation from the optimum at most

$$\varepsilon = \frac{\sum_{j \in J} \left(\lceil x_j^* \rceil - x_j^* \right)}{\sum_{j \in J} x_j^*}$$

where x_j^* is the optimal solution for the LP model.

Can we solve LP?

The Column Generation Method

Let us consider a subset $J' \subset J$ of patterns and assume that the following subproblem

$$\min \sum_{j \in J'} x_j$$

s.t.

$$\sum_{j \in J'} a_{ij} x_j \geq n_i, \quad i \in L;$$

$$x_j \geq 0, \qquad j \in J';$$

has at least one feasible solution.

Denote by x_j^* the optimal solution to this subproblem.

The Dual Problem

$$\max \sum_{i \in L} n_i \lambda_i$$

$$\sum_{i \in L} a_{ij} \lambda_i \leq 1, \quad j \in J';$$

$$\lambda_i \geq 0, \quad i \in L.$$

Denote by $\lambda_i^* \geq 0$ its optimal solution. If

$$\sum_{i \in L} a_{ij} \lambda_i^* \leq 1, \quad \text{for} \quad j \in J \setminus J'; \qquad (*)$$

then

$$\bar{x}_j = \begin{cases} x_j^*, & j \in J'; \\ 0, & j \in J \setminus J' \end{cases}$$

is the optimal solution for the LP problem.

How to Check $(*)$?

Let us consider the following knapsack problem:

$$\alpha = \max \sum_{i \in L} \lambda_i^* y_i$$

s.t.

$$\sum_{i \in L} w_i y_i \leq 1 \,; \qquad \text{(capacity of bin)}$$

$$y_i \geq 0, \ \text{integer}, \ i \in L.$$

If $\alpha \leq 1$ then $(*)$ is satisfied.

If $\alpha > 1$ then we have got a new pattern and include it in J'.

The Framework of the Method

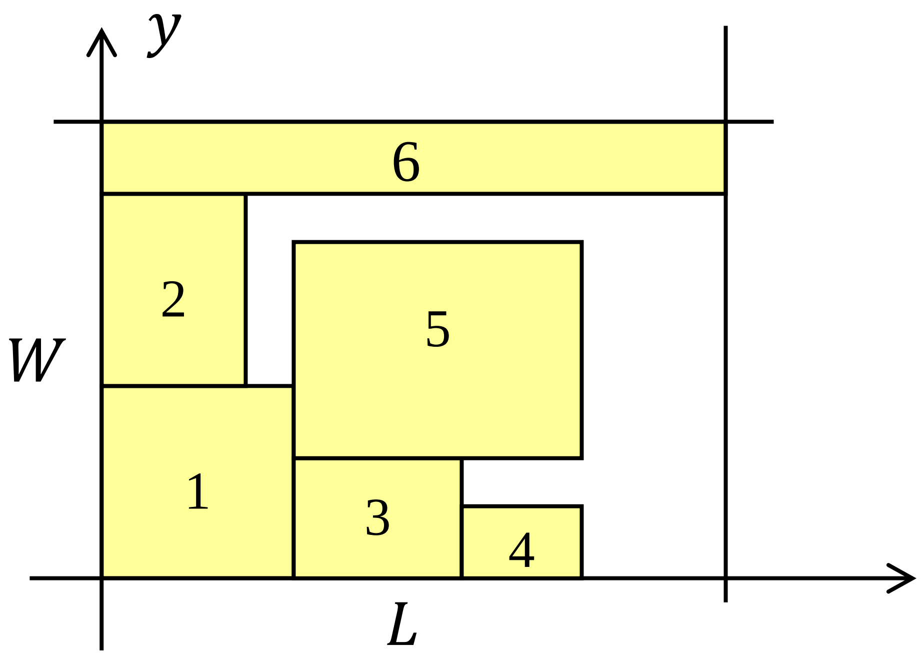
1. Select an initial subset $J' \subset J$.
2. Solve the subproblem for J' and its dual one, get x_j^*, λ_i^*.
3. Solve the knapsack problem for λ^* and compute α.
4. If $\alpha \leq 1$ then STOP.
5. Include new pattern j_0: $a_{ij_0} = y_i^*,\ i \in L,$ into subset J' and goto 2.

Surprise: As a rule, solution $x_j = \lceil x_j^* \rceil,\ j \in J$ is optimal for the bin packing problem. If it is not true, we have at most one additional bin only!

Two-Dimensional Packing Problem

Given: n rectangles with size $w_i \times l_i$, $i = 1, \ldots, n$.

Find: a packing of the rectangles into a rectangle area with minimal square.



Rotations are forbidden

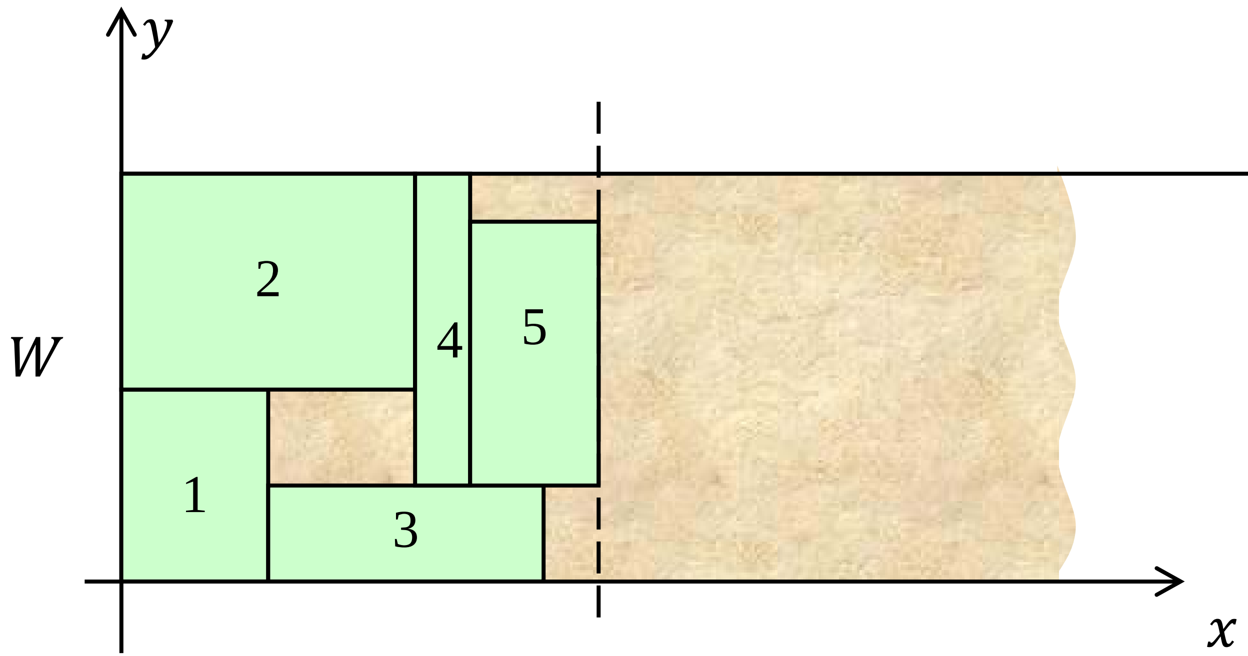
$L \times M \rightarrow \min$

It is guillotine solution.

The Strip Packing Problem

Дано: n rectangles with size $w_i \times l_i$, $i \in L$, and large strip with width W.

Find: a packing of rectangles into the strip with minimal length.



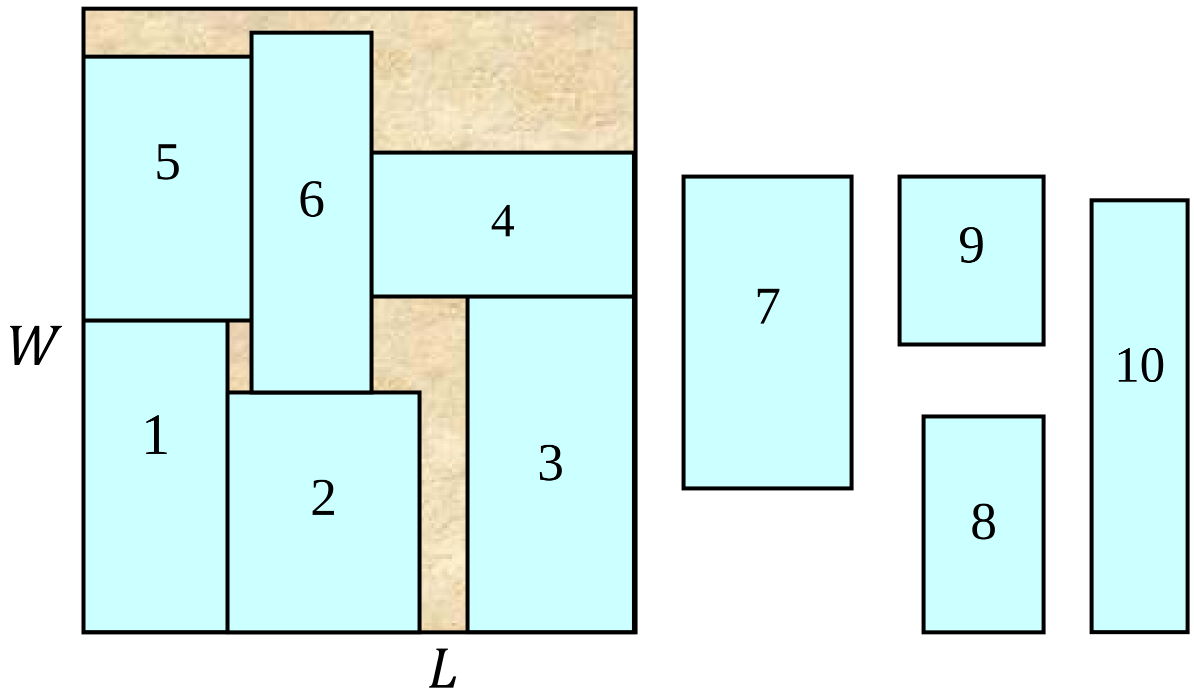
For $l_i = 1$ we have one-dimentional bin packing problem (NP-hard)

Hometask. Design a linear integer programming model for the strip packing problem (with and without $90°$ rotations).

The Two-Dimensional Knapsack Problem

Given: n rectangles with size $w_i \times l_i$, profit c_i for each rectangle, and the size of a vehicle $W \times L$.

Find: a subset of rectangles with maximal total profit which can be packed into the vehicle.



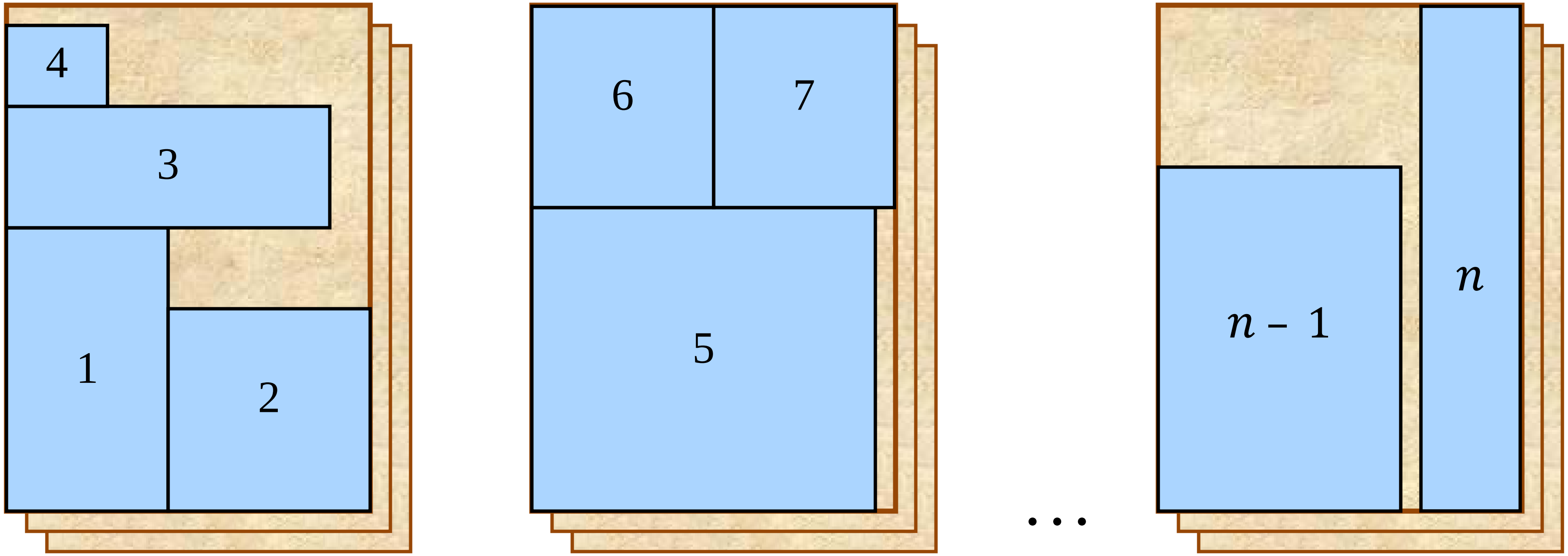
For $l_i = L$, we have the classical knapsack problem

Hometask. Design a linear integer programming model for the two-dimensional knapsack problem.

The Two-Dimensional Bin Packing Problem

Given: n rectangles with size $w_i \times l_i$ and the size of a vehicle $W \times L$.

Find: a packing all rectangles into the minimal number of vehicles.



Hometask. Design LP-based heuristic for the two-dimensional bin packing problem.