Lecture 11.

Transportation Logistics

A large part of many logistics systems involved the management of a fleet of vehicles used to serve warehouses, retailers, and/or customers.

In order to control the costs of operating the fleet, a dispatcher must continuously make decisions on how much to load an each vehicle and where to send it. These types of problems fall under the general class of ***Vehicle Routing Problem (VRP)***.

Problem Statement

The basic Vehicle Routing Problem is the single-depot Capacitated Vehicle Routing Problem (CVRP). It can be described as follows:

- A set of customers has to be served by a fleet of identical vehicles of limited capacity.
- Unlimited fleet of vehicles is initially located at a given depot.
- Each route begins at the depot, visits a subset of customers and returns to the depot without violating the capacity constraint.
- The objective is to find a set of routes for the vehicles of minimal total length.

Vehicle Routing Problem

The CVRP

Given

$J = \{0,1,\dots,n\}$ is the set of customers and $j = 0$ is the depot;

$c_{ij} \geq 0$ is the distance between i and j;

$q_i \geq 0$ is the demand of customer i;

$Q \geq 0$ is the capacity of vehicle.

Find a set of routes for the vehicles of minimal total length.

Mathematical Model

Boolean variables:

$x_{ijk} = 1$ iff vehicle k visits customer j immediately after customer i;

$y_{ik} = 1$ iff customer i is visited by vehicle k;

$$\min \sum_{i \in V} \sum_{j \in V} \sum_{k \in K} c_{ij} x_{ijk}$$

$$\sum_{k \in K} y_{ik} = \begin{cases} 1, & i = 1, \dots, n \\ m, & i = 0 \end{cases};$$

$$\sum_{i \in V} q_i \, y_{ik} \le Q, \quad k = 1, \dots, m;$$

$$\sum_{j \in V} x_{ijk} = \sum_{j \in V} x_{jik} = y_{ik}, \quad i \in J, k = 1, \dots, m;$$

$$\sum_{i,j \in S} x_{ijk} \le |S| - 1, \qquad \text{for all } S \subseteq J \setminus \{0\};$$

$$x_{ijk}, y_{ik} \in \{0,1\}.$$

Variants of the Model

- Subtour elimination constraints:

$$\sum_{i \in S} \sum_{j \in \bar{S}} x_{ijk} \geq 1, \quad S \subseteq J \setminus \{0\}, \ 2 \leq |S| \leq n-1, k = 1, \ldots, m;$$

$$u_{ik} - u_{jk} + n x_{ijk} \leq n-1, \ \ i, j \in J \setminus \{0\}, k = 1, \ldots, m.$$

- Heterogeneous fleet;
- Salary of drivers;
- Minimize the number of extra vehicle hired;
- Minimize the number of customers not served in the present period;
- Time windows for serving customers;
- $q_i > Q$: split demand for the CVRP.

A well-solved case of the CVRP

Let us assume that $Q_k = Q, k \in K$ and

$$\sum_{i \in S} q_i > Q \quad \text{for each } S \subset J \setminus \{0\}, \quad |S| \geq 3.$$

Moreover, the number of customers is even, c_{ij} is symmetric matrix and satisfies the triangle inequality: $c_{ij} + c_{jk} \geq c_{ik},\ i, j, k \in J$.

In such a case the CVRP can be solved in polynomial time by reduction to the minimum cost matching problem.

How to do that?

Dynamic Programming

Let $f(k, T)$ be the minimal cost of serving all customers in $T \subseteq J \setminus \{0\}$ using only the first k vehicles;

$v(T)$ be the minimum cost of a solution to the TSP defined by the depot and the customers in T;

$$q(T) = \sum_{i \in T} q_i.$$

Dynamic programming recursion:

$k = 1$: $\quad f(1, T) = v(T)$;

$k \geq 2$: $\quad f(k, T) = \min_{S \subset T}\{f(k-1, T-S) + v(S)\}$

for all $T \subseteq J \setminus \{0\}$ such that $q(J) - (m-k)Q \leq q(T) \leq kQ$;

$$q(T) - (k-1)Q \leq q(S) \leq Q.$$

Moreover, we must check only subsets S which satisfy the constraint

$$\frac{1}{m-k}\,q(J\setminus T) \le q(S) \le \frac{1}{k}q(T).$$

Informally, the load on route k is greater than the average load on the remaining $m-n$ routes, and less than the average load on the first $k-1$ routes.

Running time is $O(.)$?

Space requirement is $O(.)$?

Hometask. Design DP-algorithm for the Traveling Salesman Problem.

Set Covering Reformulation

Let all optimal single routes for one vehicle be indexed $r = 1, \ldots, \bar{r}$. Let the index set of customers in route r be M_r and the cost of the route (optimal TSP for M_r) be d_r. We will use $N_i = \{r \mid i \in M_r\}$ as the set of all routes which include customer $i \in J$.

Variables: $y_r = \begin{cases} 1, & \text{if route } r \text{ is in the optimal solution} \\ 0 & \text{otherwise.} \end{cases}$

$$\min \sum_{r=1}^{\bar{r}} d_r y_r$$

$$s.t. \qquad \sum_{r \in N_i} y_r = 1, \quad i \in J \setminus \{0\};$$

$$\sum_{r=1}^{\bar{r}} y_r \leq m.$$

Hometask. Consider an instance of the CVRP with 6 customer, two identical vehicles with capacity $Q = 6$ and symmetric distance matrix c_{ij}:

	0	1	2	3	4	5	6
0	–						
1	28	–					
2	21	47	–				
3	14	36	26	–			
4	17	25	37	15	–		
5	18	20	30	31	29	–	
6	22	35	20	34	39	16	–

The customer demands are $(q_1, \dots, q_6) = (2,3,1,1,2,1)$. Generate all feasible routes to this problem. Write down the Set Covering formulation. Solve this model to optimality by inspection.

Heterogeneous Fixed Sleet CVP

$f_k \geq 0$ is the fixed cost for using vehicle k;

$c_{ij}^k \geq 0$ is the traveling cost for vehicle k on the road from i to j;

$m_K > 0$ is the number of vehicles of type $k \in K$.

Variables:

$x_{ijk} = 1$ iff vehicle k visits customer j immediately after customer i;

$y_{ij} \geq 0$ is the load of a vehicle during it travel from i to j.

HFFCVRP

$$\min\left(\sum_{k\in K} f_k \sum_{j\in J\setminus\{0\}} x_{0jk} + \sum_{k\in K}\sum_{j\in J} c_{ij}^k x_{ijk}\right)$$

$$\sum_{k\in K}\sum_{i\in J} x_{ijk} = 1, \qquad j \in J \setminus \{0\};$$

$$\sum_{i\in J} x_{ijk} = \sum_{i\in J} x_{jik}, \;\; j \in J, k \in K;$$

$$\sum_{j\in J\{0\}} x_{0jk} \leq m_k, \;\; k \in K;$$

$$\sum_{i\in J} y_{ij} - \sum_{i\in V} y_{ji} = q_j, \;\; j \in J \setminus \{0\};$$

$$y_{0j} \leq \sum_{k\in K} Q_k x_{0jk}, \;\; j \in J \setminus \{0\};$$

$$y_{ij} \leq \sum_{k\in K} (Q_k - q_i) x_{ijk},$$

$$i \in J \setminus \{0\}, \;\; j \in J, \;\; i \neq j;$$

$$y_{ij} \geq 0, \;\; x_{ijk} \in \{0,1\}.$$

Subproblem for Given Sequence of Customers

Assume that we know a sequence of visiting the customers by the vehicles $\pi = (\pi_1, \pi_2, \dots, \pi_n)$. If the heterogeneous fleet of vehicles is unlimited, this subproblem of the CVRP can be solved by DP algorithm.

Hometask. Design mathematical model and DP-algorithm for this case.