Home task discussion

The SPLP is NP-hard in the strong sense.

A problem is said to be strongly NP-complete (NP-complete in strong sense), if it remains NP-complete even when all of its numerical parameters are bounded by a polynomial in the length of the input.

Node Cover Problem;

Knapsack Problem;

Maximal Clique Problem.

The traveling salesman problem

$$\min \sum_{j \in J} \sum_{i \in I} c_{ij} x_{ij}$$

s.t.

$$\sum_{i \in I} x_{ij} = 1, \;\; j \in J;$$

$$\sum_{j \in J} x_{ij} = 1, \;\; i \in I;$$

$$u_i - u_j + n x_{ij} \leq n - 1, \;\; i, j = 2, \dots, n;$$

$$x_{ij} \in \{0,1\}, \;\; u_i \geq 0, \;\; i = 1, \dots, n.$$

Lecture 2.

Basic facility location models

- The simple plant location problem
- The p-median problem
- The p-center problem
- The set covering problem

Combinatorial formulations

Proposition 1. The SPLP has the single assignment property $\left(x_{ij} = 1 \vee 0\right)$.

Proof. For each customer $j \in J$ we select a cheapest deliverer i: $x_i = 1$.

SPLP: combinatorial formulation

$$\min_{P \subseteq I} \left\{ \sum_{j \in J} \min_{i \in P} c_{ij} + \sum_{i \in P} f_i \right\}$$

The p-median problem

$$\min_{P \subset I, |P| = p} \{\sum_{j \in J} \min_{i \in P} c_{ij}\}$$

Mixed integer programming model:

$$\min \sum_{j \in J} \sum_{i \in I} c_{ij} x_{ij}$$

subject to:

$$\sum_{i \in I} x_{ij} = 1, \quad j \in J;$$

$$y_i \geq x_{ij}, \quad i \in I, j \in J;$$

$$\sum_{i \in I} y_i = p;$$

$$y_i \in \{0,1\}, \quad x_{ij} \geq 0, \quad i \in I, j \in J.$$

Has the p-median problem the Single Assignment Property?

The p-center problem

$$\min_{P \subset I, |P| = p} \{\max_{j \in J} \min_{i \in P} c_{ij}\}$$

Mixed integer linear programming model:

$$\min D$$

subject to:

$$D \geq c_{ij} x_{ij}, \;\; i \in I, j \in J;$$

$$\sum_{i \in I} x_{ij} = 1, \;\; j \in J;$$

$$y_i \geq x_{ij}, \;\; i \in I, j \in J;$$

$$\sum_{i \in I} y_i = p;$$

$$y_i \in \{0,1\}, \;\; x_{ij} \geq 0, \;\;\; i \in I, j \in J.$$

How about relaxation $x_{ij} \in \{0,1\} \quad \Rightarrow \quad 0 \leq x_{ij} \leq 1$?

Why we discuss the relaxations?

Suppose that we have an algorithm for the p-median problem.

- Can we solve the p-center problem by using this algorithm?
- What is computational complexity of your approach?
- Can we say that the p-center problem is NP-hard?

The set covering problem

Let $A = \{a_{ij}\}$ be an 0–1 matrix:

$$a_{ij} = \begin{cases} 1 & \text{if facility } i \text{ can service customer } j \\ 0 & \text{otherwise} \end{cases}$$

A subset $P \subseteq I$ of facilities defines a cover of J if $\sum_{i \in P} a_{ij} \geq 1$ for all $j \in J$.

The set covering problem is to find a cover of minimal cost:

$$\min \sum_{i \in I} f_i y_i$$

subject to

$$\sum_{i \in I} y_i a_{ij} \geq 1, \quad j \in J;$$

$$y_i \in \{0,1\}, \quad i \in I.$$

Can we claim that this problem is NP-hard?

Pseudo-Boolean reformulations

For a vector $g = (g_1, \dots, g_m)$ with ranking $g_{i_1} \le g_{i_2} \le \dots \le g_{i_m}$ we introduce a vector $\Delta g = (\Delta g_0, \dots, \Delta g_m)$ in the following way:

$$\Delta g_0 = g_{i_1}$$

$$\Delta g_l = g_{i_{l+1}} - g_{i_l}, \quad 1 \le l < m;$$

$$\Delta g_m = g_{i_m}.$$

Lemma. For each 0-1 vector $z = (z_1, \dots, z_m)$, $z \ne (1, \dots, 1)$ we have

$$\min_{i | z_i = 0} g_i = \Delta g_0 + \sum_{l=1}^{m-1} \Delta g_l z_{i_1} \dots z_{i_l}.$$

Example. $g =$ (10,8,5,7,1,9); $\Delta g =$ (1, 4=5-1, 2=7-5, 1=8-7, 1=10-9);

$z =$ (1,1,0,0,1,0), $z =$ (1,1,0,0,0,0), $z =$ (1,1,1,1,1,0).

Pseudo-Boolean reformulation for SPLP

Let the ranking for column j of the matrix (c_{ij}) be

$$c_{i_1^j j} \leq c_{i_2^j j} \leq \cdots \leq c_{i_m^j j}$$

Using Lemma, we can get a pseudo-boolean function for the SPLP:

$$z_i = 1 - y_i, \quad i \in I;$$

$$\min_z \left\{ \sum_{i \in I} f_i (1 - z_i) + \sum_{j \in J} \sum_{l=0}^{m-1} \Delta c_{lj} z_{i_1^j} \ldots z_{i_l^j} \right\}.$$

Example. $I = J = \{1,2,3\}$; $f_i = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}$; $c_{ij} = \begin{pmatrix} 0 & 3 & 10 \\ 5 & 0 & 0 \\ 10 & 20 & 7 \end{pmatrix}$;

We get the pseudo-boolean function: $b(z) = 10(1 - z_1) + 10(1 - z_2) +$
$+10(1 - z_3) + (5z_1 + 5z_1 z_2) + (3z_2 + 17z_1 z_2) + (7z_2 + 3z_2 z_3) =$
$= 15 + 5(1 - z_1) + 0(1 - z_2) + 10(1 - z_3) + 22z_1 z_2 + 3z_2 z_3.$

Correspondence “many to one”

Theorem. The minimization problem for the pseudo-boolean function $b(z)$ for $z \neq (1, \dots, 1)$ and the SPLP are equivalent. For optimal solutions z^*, P^* of these problems we have

$$F(P^*) = b(z^*) \text{ and } z_i^* = 0 \Leftrightarrow i \in P^* \text{ for all } i \in I.$$

Example. Optimal solutions for these instances are the same or not?

$$f = \begin{pmatrix} 5 \\ 0 \\ 10 \end{pmatrix} c_{ij} = \begin{pmatrix} 0 & 3 \\ 0 & 0 \\ 22 & 0 \end{pmatrix} \text{ and } f = \begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix} c_{ij} = \begin{pmatrix} 4 & 1 \\ 0 & 0 \\ 1 & 23 \end{pmatrix}$$

Reduction of the set J

Theorem. For the minimization problem of the pseudo-boolean function $b(z)$ with positive coefficients in the nonlinear terms, the equivalent instance of the SPLP with minimal number of customers can be found in polynomial time from n and m.

Sketch of proof. Consider a pseudo-boolean function $b(z)$ define by

$$b(z) = \sum_{i \in I} \alpha_i (1 - z_i) + \sum_{l \in L} \beta_l \prod_{i \in I_l} z_i,$$

where $\beta_l > 0$ and $I_l \subset I$ for all $l \in L$.

The family of subset $\{I_l\}_{l \in L}$ of the set I with order relation $I_{l'} < I_{l''} \Leftrightarrow I_{l'} \subset I_{l''}$ forms a partially ordered set (poset). An arbitrary sequence of subsets $I_{l_1} < \cdots < I_{l_k}$ is called a chain. An arbitrary partition of the family $\{I_l\}_{l \in L}$ into nonoverlapping chains induces a matrix (c_{ij}) for the UFLP. Each element of the partition corresponds a user. The requirement to find an instance of UFLP with a minimal number of users is equivalent to finding a partition of the poset into the minimal number of nonoverlapping chains. This is a well-known problem which can be solved in polynomial time. ■

Example:

Function $b(z) = z_1z_2 + z_1z_3 + z_1z_4 + z_1z_5 + z_3z_4 + z_3z_5 + z_4z_5 + z_1z_2z_3 + z_1z_3z_4 + z_1z_4z_5 + z_1z_2z_3z_4 + z_2z_3z_4z_5 + z_1z_3z_4z_5 - 10z_1 - 10\, z_2 - 5z_3 - 5z_4 - 6\, z_5$.

Generate an equivalent SPLP instance: $I = \{1, \dots, 5\}$, $J = \{1, \dots\ ?\}$.

What can we say about the integrality gap for the minimization problem for the pseudo-boolean function?

Paradox

Integrality gap for the SPLP can be arbitrary close to 1 (see lecture 1) but for the minimization problem $b(z)$ **the gap equals 0**.

Theorem. The set of optimal solutions of the minimization problem for arbitrary pseudo-boolean function with continuous variables contains a pure integer solution.

Proof. For boolean variable z, we have $z = z^2$ $(0 = 0^2, 1 = 1^2)$. So, we may assume that pseudo-boolean function has not $z^2, z^3, \ldots$. All nonlinear terms are product of different variables.

Let z^* be the optimal solution for $b(z)$ and assume that $z_1^* \neq 0 \vee 1$. We can rewrite $b(z)$:

$$b(z) = b_1(z_2, \ldots, z_m) + z_1 b_2(z_2, \ldots, z_m).$$

If $b_2(z_2^*, \ldots z_m^*)$ is positive, then we can improve our optimal solution by decreasing z_1^* to 0. If it is negative, we put $z_1^* = 1$. Hence, $b_2(z_2^*, \ldots, z_M^*) = 0$ and we can put $z_1^* = 1 \vee 0$. ■

Can it helps us to solve the problem?

Boolean linear and nonlinear optimization

Boolean linear problem:

$$\text{(BL)}: \quad \begin{aligned} &\min cx \\ &Ax \geq b; \\ &x \in B^n. \end{aligned}$$

It is NP-hard problem (SPLP).

Note that $y \in \{0,1\} \Leftrightarrow y(1-y) = 0.$

Hence, the BL is equivalent to

$$\text{(NLP)}: \quad \begin{aligned} &\min cx \\ &Ax \geq b; \\ &x_i(1-x_i) = 0, \quad i = 1, \dots, n; \\ &x \in R^n. \end{aligned}$$

Hometask 1

In the SPLP we have $I = J = \{\text{Altus, Ardmore, Bartlesville, Duncan, Edmond, Enid}\}$, $n = m = 6$

$$c_{ij} = \begin{bmatrix} 0 & 169 & 291 & 88 & 153 & 208 \\ 169 & 0 & 248 & 75 & 112 & 199 \\ 291 & 248 & 0 & 231 & 146 & 132 \\ 88 & 75 & 231 & 0 & 93 & 137 \\ 153 & 112 & 146 & 93 & 0 & 88 \\ 208 & 199 & 132 & 137 & 88 & 0 \end{bmatrix} \qquad f_i = \begin{pmatrix} 150 \\ 150 \\ 150 \\ 100 \\ 100 \\ 100 \end{pmatrix}$$

Find the equivalent minimization problem for a pseudo-boolean function.

Hometask 2

For the following instance of the set covering problem: $I = \{1, \dots, 5\}$, $J = \{1, \dots, 4\}$, $f_i = 1$ for $i \in I$, and

$$a_{ij} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

find the equivalent minimization problem for a pseudo-boolean function.

Questions

1. Suppose that we have an algorithm for the set covering problem. Can we solve the p-center problem using this algorithm?
2. The set covering problem is NP-hard in the strong sense. If the previous reduction is correct, can we claim that the p-center problem is NP-hard in the strong sense?
3. The minimization problem for arbitrary pseudo-boolean function is NP-hard in the strong sense or not?