Lecture 3.

Hardness of Approximation

*An algorithm A is called a **ρ-approximation algorithm** for a minimization (maximization) problem Π, if for every instance of Π it delivers a feasible solution of value at most (at least) ρf^*, where f^* denotes the optimal value for the instance.*

The class of optimization problems for which a constant approximation algorithm that runs in polynomial time exists is called the class of ***APX-problem***.

APX-problems: maximum knapsack, minimum bin-packing, maximum satisfiability …

How about SPLP, p-median, set covering, p-center?

Theorem. The p-median minimization problem with a constant approximation ratio ρ is NP-hard for any $\rho > 1$.

Proof. We will show that a polynomial time ρ-approximation algorithm for the p-median problem implies an exact polynomial time algorithm for the Node Cover Problem.

Given an instance of the NCP: graph $G = (V, E)$ and an integer threshold k. We create an instance of the p-median problem: $I = V,\ J = E,\ p = k$ and

$$c_{ij} = \begin{cases} 1 & \text{if edge } e_j \text{ is incident to node } i \\ \rho|E| & \text{otherwise} \end{cases}$$

Now G contains a node cover of size k, if and only if $Opt_{p-\text{median}} = |E|$. Hence, a ρ-approximation algorithm yields a solution with $f_A \leq \rho|E|$. This implies that G contains a node cover of size k since any feasible solution that does not cover all of the edges has cost of at least $|E| - 1 + \rho|E| > \rho|E|$. ■

In other words, the p-median minimization problem does not belong to the class APX if P$\neq$ NP.

- *Can we design a ρ-approximation algorithm where ρ is a function of $|I|$ and $|J|$?*
- *How about the p-center minimization problem?*
- *What can we say about the Euclidean case?*

The p-Median Maximization Problem

$$\max_{P \subset I, |P|=p} \left\{ \sum_{j \in J} \max_{i \in P} c_{ij} \right\}.$$

Greedy heuristic opens a new facility at each step and selects the most profitable facility.

Theorem. $\frac{f_{greedy}}{Opt} \geq \left(1 - \left(\frac{p-1}{p}\right)^p\right) \geq \frac{e-1}{e} \approx \ 0.63$. Moreover, for each p there is an instance for which the bound is tight; that is

$$\rho = 1 - \left(\frac{p-1}{p}\right)^p.$$

In other words, the p-median maximization problem belongs to the class APX.

The Set Covering Problem

Greedy algorithm for case $f_i = f$ for all $i \in I$.

Put $J_i = \{j \in J | a_{ij} = 1\}$ for all $i \in I$.

1. $J' \leftarrow J, S \leftarrow \emptyset$
2. **While** $J' \neq \emptyset$
3. **do** select a facility i that maximizes $|J_i \cap J'|$
4. $J' \leftarrow J' \setminus J_i$
5. $S \leftarrow S \cup \{i\}$
6. Return S

Can we say that this algorithm is a polynomial?

How we should modify it for general case $(f_i \neq f)$ *?*

What is your feeling about its approximation ratio? (You know it!)

Theorem. For the Set Covering Problem the greedy heuristic is ρ-approximation algorithm with $\rho = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{|J|}$.

Bed news: There is a small constant γ such that if there is a ρ-approximation algorithm for the SCP with $\rho = \gamma(1 + \frac{1}{2} + \cdots + \frac{1}{|J|})$ then NP $\subseteq$ ZTIME $\left(n^{O(\log\log n)}\right)$.

What can we say about the SPLP?

How about the maximization version of the SPLP?

Hometask

Show how to implement greedy algorithm in such a way that its runs in time $O(mn),\ n = |J|,\ m = |I|$.