Lecture 4.

Lower and Upper Bounds for Global Optimum

Consider an integer linear program

$$z_{IP} = \max\{\, cx \mid Ax \le b,\ \ x \in Z_+^n \,\}$$

How we can get an upper bound for z_{IP}?

How we can get an lower bound for z_{IP}?

Lagrangian Relaxation

We rewrite this program as

$$z_{IP} = \max cx$$

$$A^1 x \leq b^1 \quad \text{(complicating constraints)}$$

$$A^2 x \leq b^2 \quad \text{(nice constraints)}$$

$$x \in Z_+^n$$

If we drop the complicating constraints, then we obtain a relaxation that is easier to solve than the original problem.

We assume that $Q = \{x \in Z_+^n \mid A^2 x \leq b^2\} \neq \emptyset$.

Now for any nonnegative vector λ we consider the problem

$$\text{LR}(\lambda): \qquad z_{LR}(\lambda) = \max\{z(\lambda, x) \mid x \in Q\}$$

where $z(\lambda, x) = cx + \lambda(b^1 - A^1 x)$.

The problem $\text{LR}(\lambda)$ is called the *Lagrangian relaxation* of $IP(Q)$ with respect to $A^1 x \leq b^1$.

Theorem. $z_{LR}(\lambda) \geq z_{IP}$ for all $\lambda \geq 0$.

Proof. Let us consider a feasible solution x of $IP(Q)$. Note that x is feasible for $\text{LR}(\lambda)$ as well. Moreover, $\lambda(b^1 - Ax^1) \geq 0$ for x because $\lambda \geq 0$ and $A^1 x \leq b^1$.

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Lagrangian Dual

The least upper bound available from the infinite family of relaxations $\{\mathrm{LR}(\lambda)\}_{\lambda\geq 0}$ is $z_{LR}(\lambda^*)$, where λ^* is an optimal solution to

$$\text{LD:} \qquad z_{LD} = \min_{\lambda\geq 0} z_{LR}(\lambda)$$

Problem LD is called the *Lagrangian dual* of $IP(Q)$ with respect to the constraints $A^1x \leq b^1$.

Theorem. $z_{LD} = \max\{cx \,|A^1x \leq b^1, x \in \mathrm{conv}(Q)\}$ where $\mathrm{conv}(Q)$ is a rational polyhedron, $\mathrm{conv}(Q)$ is a minimal convex polyhedron which contains all discrete points from $Q = \{x \in Z_+^n \,|A^2x \leq b^2\}$.

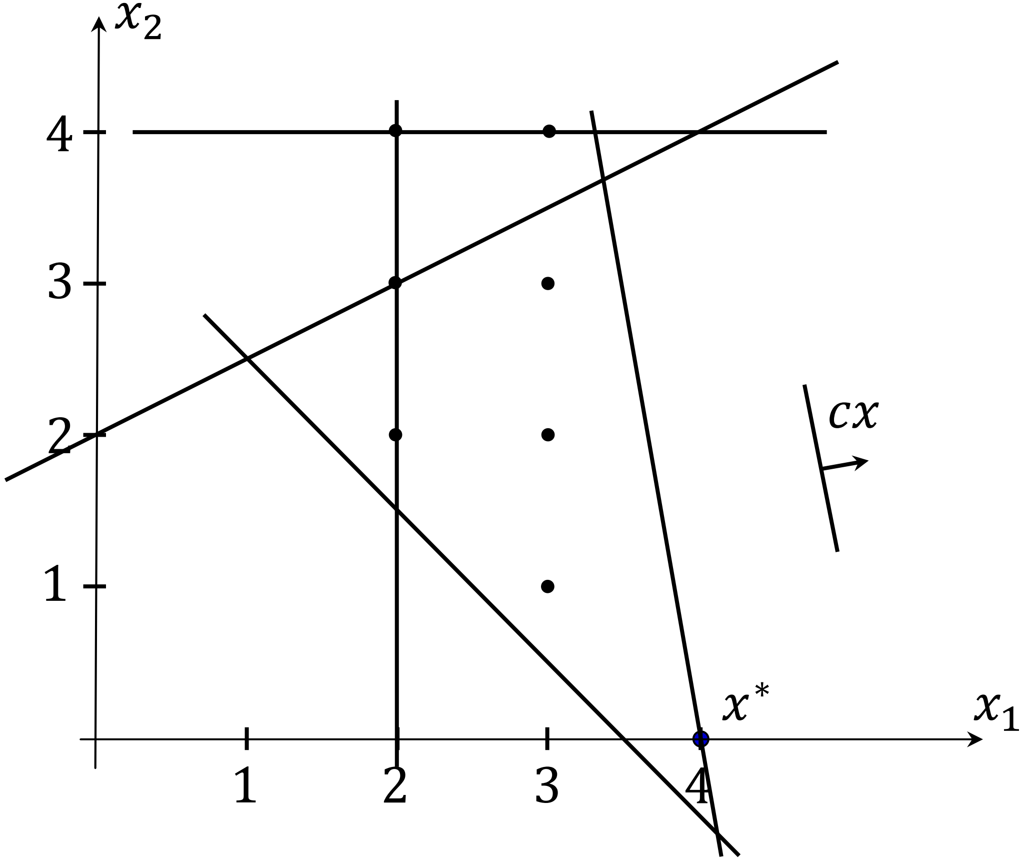
Example 1

$$\max 7x_1 + 2x_2$$

s.t.

$$-x_1 + 2x_2 \le 4 \qquad (A^1 x \le b^1)$$

$$\left.\begin{aligned} 5x_1 + x_2 &\le 20 \\ -2x_1 - 2x_2 &\le -7 \\ -x_1 &\le -2 \\ x_2 &\le 4 \\ x &\in Z_+^2 \end{aligned}\right\} \qquad Q = \{x \in Z_+^2 \mid A^2 x \le b^2\}$$



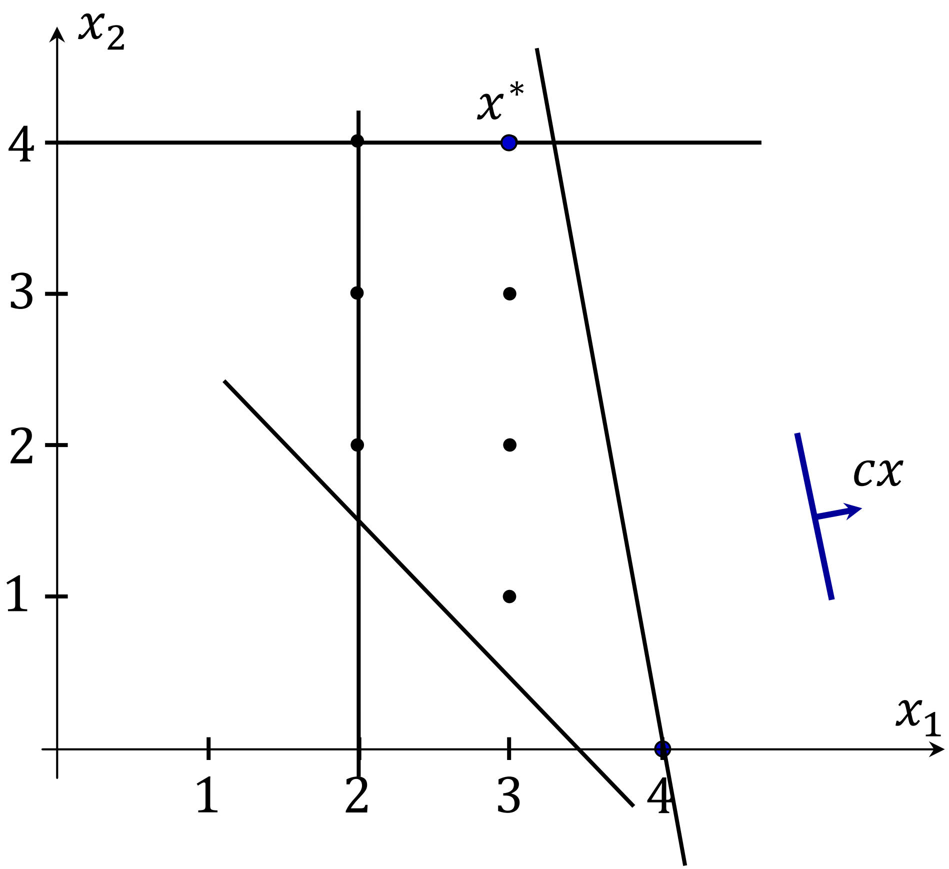
$z_{IP} = 28$

$x^* = (4,0)$;

All feasible points: $(2,2), (2,3), (3,1), (3,2), (3,3), (4,0)$

If we drop the constraint $-x_1 + 2x_2 \leq 4$ then we get the relaxed problem with optimal value 29. The set Q is finite and contains 8 points:

$$(x^1, x^2, \dots, x^8) = \{(2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,0)\}$$



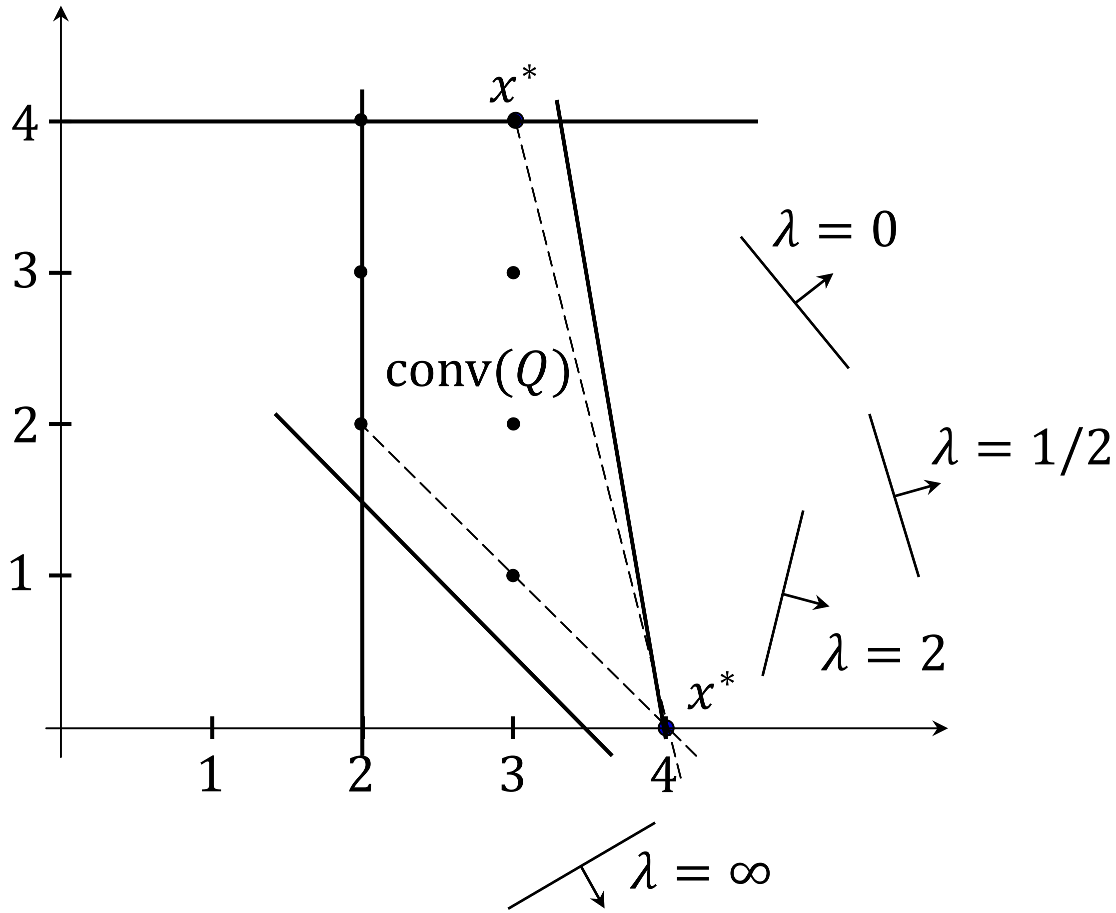
Lagrangian relaxation with respect to constraint $-x_1 + 2x_2 \leq 4$ is

$$z_{LR}(\lambda) = \max_{x \in Q}\{7x_1 + 2x_2 + \lambda(4 + x_1 - 2x_2)\} =$$

$$= \max\{(7+\lambda)x_1 + (2 - 2\lambda)x_2 + 4\lambda\}$$

s.t.

$$\begin{aligned} 5x_1 + \ x_2 &\leq 2; \\ -2x_1 - \ 2x_2 &\leq -7; \\ -x_1 \qquad\quad &\leq -2; \\ x_2 &\leq 4; \\ x &\in Z_+^2. \end{aligned}$$



$$z_{LD} = 28\,{}^{8}/_{9}, \qquad \lambda^* = {}^{1}/_{9}, \qquad x^* = x^7 \; (x^* = x^8)$$

How to solve the dual problem?

Let $x^*(\lambda^0)$ be the optimal solution for the Lagrangian relaxation $LR(\lambda^0)$.

Then

$$S = b^1 - A^1 x^*(\lambda^0)$$

is the subgradient of function $z_{LR(\lambda)}$ at the point $\lambda = \lambda^0$. We put

$$\lambda^1 = \lambda^0 + \beta^0 S,$$

where β is a suitable scalar coefficient.

If $\beta^k \to 0,\ \sum_{k=1}^{\infty} \beta^k = \infty$ then $z_{LR(\lambda)} \to z_{LD}$.

Example 2

The assignment problem with budget constraint

The is a set of n jobs to be assigned to a set of n workers, $N = \{1, \ldots, n\}$.

c_{ij} is the value of assigning worker i to job j;

t_{ij} is the cost of training worker i to do job j;

b is training budget.

We wish to maximize the total value of the assignment subject to the budget constraint.

Optimization model:

$$\max \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij}$$

s.t.

$$\sum_{i \in N} x_{ij} = 1, \quad j \in N; \tag{1}$$

$$\sum_{j \in N} x_{ij} = 1, \quad i \in N; \tag{2}$$

$$\sum_{i \in N} \sum_{j \in N} t_{ij} x_{ij} \leq b; \tag{3}$$

$$x_{ij} \in \{0,1\}, \qquad i, j \in N.$$

How to choose a Lagrangian relaxation?

Let us consider 4 variants:

1. Lagrangian relaxation with respect to (3):

$LR_1(\lambda)$:
$$\max \sum_{i \in N} \sum_{j \in N} (c_{ij} - \lambda t_{ij}) x_{ij} + \lambda b$$

s.t. (1), (2)

It is well-known assignment problem, integrality gap is 0, the linear programming relaxation has an integer optimal solution.

Hence,

$$z_{LP} = z_{LD}^1 \geq z_{IP}.$$

2. Lagrangian relaxation with respect to (1) and (2)

$LR_2(u,v)$:

$$\max \sum_{i \in N} \sum_{j \in N} (c_{ij} - u_i - v_j) x_{ij} + \sum_{i \in N} u_i + \sum_{j \in N} v_j$$

s.t. (3).

It is well-known knapsack problem, integrality gap is nonnegative (often positive). Hence,

$$z_{LP} = z_{LD}^1 \geq z_{LD}^2 \geq z_{IP}.$$

We can get better upper bound than in previous case.

3. Lagrangian relaxation with respect to (2)

$LR_3(u)$:

$$\max \sum_{i \in N} \sum_{j \in N} (c_{ij} - u_i) x_{ij} + \sum_{i \in N} u_i$$

s.t. (1), (3).

It is well-known multiple-choice knapsack problem. Integrality gap is nonnegative (often positive). Moreover, each feasible solution for LR_3 is feasible in the LR_2. Hence,

$$z_{LP} = z_{LD}^1 \geq z_{LD}^2 \geq z_{LD}^3 \geq z_{IP}.$$

We can improve the previous upper bound.

4. Lagrangian relaxation with respect to (2) and (3)

$LR_4(u,\lambda)$:

$$\max \sum_{i\in N}\sum_{j\in N}(c_{ij}-u_i-\lambda t_{ij})x_{ij}+\sum_{i\in N}u_i+\lambda b$$

s.t. (1).

It is trivial to solve. For each j we maximize $c_{ij}-u_i-\lambda t_{ij}$ and the corresponding x_{ij} is set to 1. Hence, the gap is 0 and

$$z_{LP}=z_{LD}^4=z_{LD}^1\geq z_{LD}^2\geq z\geq_{LD}^3\geq z_{IP}.$$

Should we relax (1), (2), (3) at the same time or (1),(3)?

Lagrangian Relaxation for the SPLP

$$\min \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{i \in I} f_i y_i$$

s.t.

$$\sum_{i \in I} x_{ij} \geq 1, \quad j \in J;$$

$$y_i \geq x_{ij}, \quad j \in J, i \in I;$$

$$y_i, x_{ij} \in \{0,1\}.$$

Two ways to relax the problem.

What is the best one?

Relaxation 1

$LR1(\lambda)$: $$\min \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{i\in I} f_i y_i + \sum_{j\in J} \lambda_j \left(1 - \sum_{i\in I} x_{ij}\right)$$

s.t. $$y_i \geq x_{ij}, \qquad i \in I, j \in J;$$

$$y_i,\ x_{ij} \in \{0,1\}, \qquad i \in I, j \in J;$$

$$z_{LR1(\lambda)} = \min \sum_{i\in I}\sum_{j\in J} (c_{ij} - \lambda_j)x_{ij} + \sum_{i\in I} f_i y_i + \sum_{j\in J} \lambda_j\,;$$

Can we solve this problem in polynomial time?

Relaxation 2

$LR2(\gamma)$: $$\min \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{i\in I} f_i y_i + \sum_{i\in I}\sum_{j\in J}\gamma_{ij}(x_{ij} - y_i)$$

s.t.

$$\sum_{i\in I} x_{ij} \geq 1, \quad j \in J;$$

$$y_i,\ x_{ij} \in \{0,1\}, \qquad i \in I, j \in J;$$

$$z_{LR2(\lambda)} = \min \sum_{i\in I}\sum_{j\in J}(c_{ij} + \gamma_{ij})x_{ij} + \sum_{i\in I}\left(f_i - \sum_{j\in J}\gamma_{ij}\right)y_i\,;$$

Can we find $LR2(\gamma)$ in polynomial time?

Hometask 1. The Capacitated Facility Location Problem

$$\min\left\{\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{i\in I} f_i y_i\right\}$$

s.t.

$$\sum_{i\in I} x_{ij} = 1, \;\; j \in J;$$

$$y_i \geq x_{ij}, \;\; i \in I, j \in J;$$

$$\sum_{i\in I} y_i = p;$$

$$\sum_{j\in J} q_j x_{ij} \leq Q_i y_i, \;\; i \in I;$$

$$y_i \in \{0,1\}, \;\; x_{ij} \geq 0, \;\; i \in I, j \in J.$$

Can we solve the Lagrangian relaxation problem with respect to $\sum_{i\in I} x_{ij} = 1, j \in J,$ in polynomial time?

Hometask 2.

A company has two plants and three warehouses. The first plant can supply at most 100 units and the second and most 200 units of the same product. The sales potential at the first warehouse is 150, at the second warehouse 200, and at the third 350. The sales revenues per unit at the three warehouses are $\$12$ at the first, $\$14$ at the second, and $\$15$ at the third. The cost of manufacturing one unit at the plant i and shipping it to warehouse j is given in table. The company wishes to determine how many units should be shipped from each plant to each warehouse so as to maximize profit.

Table.

From plant	To warehouse ($\$$)		
	1	**2**	**3**
1	8	10	12
2	7	9	11

Create a mathematical model and solve it by Exel (*Поиск решения*).