# Lecture 5. Lagrangian Heuristics

We consider the CFLP

$$\min\left\{\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{i\in I} f_i y_i\right\}$$

s.t.

$$\sum_{i\in I} x_{ij} = 1, \quad j \in J;$$

$$y_i \geq x_{ij}, \quad i \in I, j \in J;$$

$$\sum_{i\in I} y_i = p;$$

$$\sum_{j\in J} q_j x_{ij} \leq Q_i y_i, \quad i \in I;$$

$$y_i \in \{0,1\}, \quad x_{ij} \geq 0, \quad i \in I, j \in J.$$

The first constraint is removed and included into the objective function with multipliers $\lambda_j, \ j \in J$.

We obtain the following problem:

$$\min\left\{\sum_{i\in I}\sum_{j\in J}(c_{ij}-\lambda_j)x_{ij}+\sum_{i\in I}f_i y_i\right\}+\sum_{j\in J}\lambda_j$$

$LR(\lambda)$:

$$y_i \geq x_{ij},\ \ i\in I, j\in J;$$

$$\sum_{i\in I} y_i = p;$$

$$\sum_{i\in I} q_i x_{ij} \leq Q_i y_i,\ \ i\in I;$$

$$y_i\in\{0,1\},\ \ x_{ij}\geq 0,\ \ i\in I, j\in J.$$

*How to solve this problem for given* $\lambda_j$ *?*

Note that if $y_i = 0$ then $x_{ij} = 0$ for all $j \in J$.

If $y_i = 1$ then we have the knapsack subproblem to determine $x_{ij}$:

$$\min \sum_{j \in J} (c_{ij} - \lambda_j) x_{ij}$$

s.t.

$$\sum_{j \in J} q_j x_{ij} \leq Q_i;$$

$$0 \leq x_{ij} \leq 1, \;\; j \in J.$$

*Can we find an optimal solution in polynomial time?* $T = O(?),\ \Pi = O(?)$.

Let $x^*_{ij}$ be the optimal solution of the knapsack problem and let

$$a_i(\lambda) = \sum_{j \in J} (c_{ij} - \lambda_j) x^*_{ij}.$$

We can rewrite the $LR(\lambda)$ as follows:

$$\min \sum_{i \in I} \big(f_i + a_i(\lambda)\big) y_i$$

$LR(\lambda)$: s.t.

$$\sum_{i \in I} y_i = p;$$

$$y_i \in \{0,1\}.$$

*Can we find an optimal solution $y^*_i$ in polynomial time? $T = O(?\,)$, $\Pi = O(?\,)$.*

Let $y_i^*, x_{ij}^*$ be the optimal solution of the $LR(\lambda)$.

*Is it feasible solution for the CFLP?*

*Would we solve the problem if we replace $\sum y_i = p$ by $\sum y_i \leq p$ ?*

*How we can guarantee that solution $y_i^*$, $x_{ij}^*$ can be transform to a feasible one?*

Let us introduce an additional constraint in the $LR(\lambda)$:

$$\min \sum_{i \in I} (f_i + a_i(\lambda)) y_i + \sum_{j \in J} \lambda_j$$

$$\sum_{i \in I} y_i = p;$$

$$\sum_{i \in I} Q_i y_i \geq \sum_{j \in J} q_j \, ;$$

$$y_i \in \{0,1\}.$$

Can we claim that optimal value of this problem is a lover bound for the CFLP?

Is it NP-hard?

**Theorem 5.1.** Let $y_i^*, x_{ij}^*$ be the optimal of the $LR(\lambda)$ and

$$b_j \triangleq \sum_{i \in I} x_{ij}^*, \quad j \in J.$$

Then $y_i^*, x_{ij}^*$ is the optimal solution of the CFLP where constraint

$$\sum_{i \in I} x_{ij} = 1, \quad j \in J \quad \text{is replaced by} \quad \sum_{i \in I} x_{ij} = b_j, \quad j \in J.$$

*How to prove it?*

## Lagrangian Heuristics

Let us consider a sequence of Lagrangian multipliers $\{\lambda^k\}$

$$\lambda_j^k = \lambda_j^{k-1} + \beta(1 - \sum_{i \in I} x_{ij}^*(\lambda^{k-1})\,).$$

For each $\lambda^k$ we have $y_i^*(\lambda^k)$, $x_{ij}^*(\lambda^k)$ and solve the linear program:

$$\min_{x_{ij} \geq 0} \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$$

s.t.

$$\sum_{i \in I} x_{ij} = 1, \;\; j \in J;$$

$$y_i^* \geq x_{ij}, \;\; i \in I, j \in J;$$

$$\sum_{j \in J} q_j x_{ij} \leq Q_i y_i^*, \;\; i \in I;$$

If $\bar{x}_{ij}$ is an optimal solution, then $y_i^*, \bar{x}_{ij}$ is feasible one for the CFLP.

## Hometask 1.

In the CFLP we have $I = J = \{\text{Altus, Ardmore, Bartlesville, Duncan, Edmond, Enid}\}$, $n = m = 6$

$$c_{ij} = \begin{bmatrix} 0 & 169 & 291 & 88 & 153 & 208 \\ 169 & 0 & 248 & 75 & 112 & 199 \\ 291 & 248 & 0 & 231 & 146 & 132 \\ 88 & 75 & 231 & 0 & 93 & 137 \\ 153 & 112 & 146 & 93 & 0 & 88 \\ 208 & 199 & 132 & 137 & 88 & 0 \end{bmatrix} \qquad f_i = \begin{pmatrix} 150 \\ 150 \\ 150 \\ 100 \\ 100 \\ 100 \end{pmatrix}$$

Apply the Lagrangian heuristic for the CFLP with

$$\lambda_j \equiv 200, \quad q_j = (5, 7, 7, 6, 7, 5); \quad Q_i \equiv 15, \quad p = 3.$$

*How far this heuristic solution from the global optimum?*