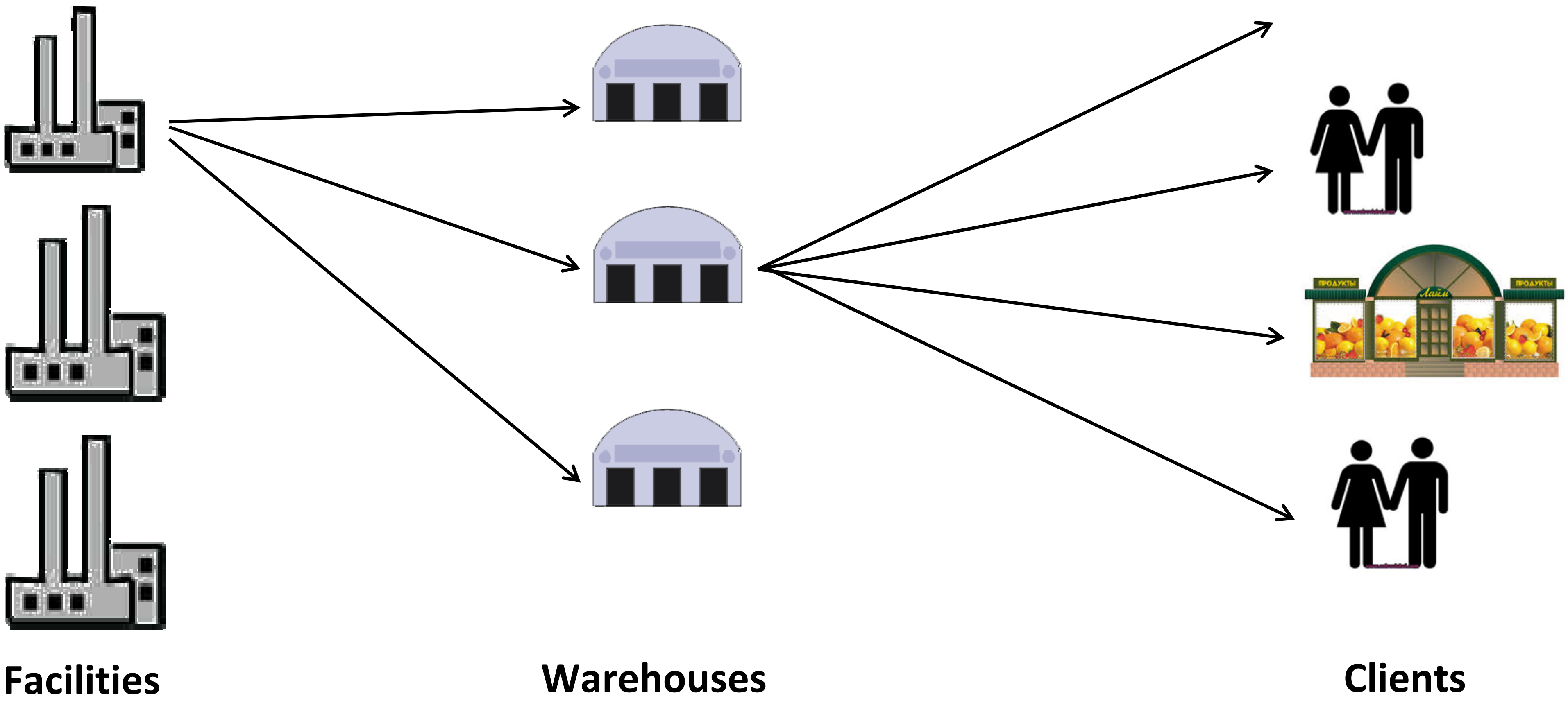
Lecture 6. Two-Echelon Capacitated Facility Location Problem



We wish to minimize the total cost of opening facilities and warehouses and transportation cost for serving all clients.

Problem Statement

Given: V is the set of facilities;

U is the set of warehouses;

J is the set of clients;

$K = \{(i_1, i_2): i_1 \in V, i_2 \in U\}$ is the set of pairs (facility, warehouse);

$K_i = \{k \in K | k = (i, i') \vee (i', i)\}$ is subset of pairs for given i;

f_i is the fixed cost of opening facility i;

g_i is the fixed cost of opening warehouse i;

c_{kj} is the transportation cost for serving client j by pair k;

q_j is the demand of client j;

Q_k is the capacity of pair k.

Find: the minimal total cost and optimal subset of facilities and warehouses.

Mathematical Model

$$\min\left(\sum_{i\in V} f_i y_i + \sum_{i\in V} g_i z_i + \sum_{k\in K}\sum_{j\in J} c_{kj}x_{kj}\right)$$

s.t.

$$\sum_{k\in K} x_{kj} = 1, \quad j \in J;$$

$$y_i \geq t_k, \quad k \in K_i,\ i \in V;$$

$$z_i \geq t_k, \quad k \in K_i,\ i \in U;$$

$$t_k \geq x_{kj}, \quad k \in K, j \in J;$$

$$\sum_{j\in J} q_j x_{kj} \leq Q_k, \quad k \in K;$$

$$y_i, z_i, t_k \in \{0,1\}, \qquad x_{kj} \geq 0.$$

Is this problem NP-hard? Can we include the production cost into the model?

Can we include transportation costs between facilities and warehouses?

Can we include capacity constraints for facilities and warehouses?

Can we improve the linear programming relaxation

replacing $\begin{cases} y_i \geq t_k, & k \in K_i, \quad i \in V \\ z_i \geq t_k, & k \in K_i, \quad i \in U \end{cases}$ by $\begin{cases} y_i \geq \sum_{k \in K_i} x_{kj}, & i \in V, j \in J \\ z_i \geq \sum_{k \in K_i} x_{kj}, & i \in U, j \in J \end{cases}$

Theorem 5.2. The multi-echelon uncapacitated facility location problem is equivalent to the minimization problem for pseudo-boolean function

$$B(z) = \sum_{i \in I} f_i \left(1 - \prod_{k \in K_i} z_k \right) + \sum_{j \in J} \sum_{l=1}^{|K|-1} \Delta c_{lj} z_{k_1^j} \dots z_{k_l^j},$$

where (f_i) is the fixed cost of facilities, warehouses, distribution centers and others.

LR_1:

$$\min\left\{\sum_{i\in V} f_i y_i + \sum_{i\in U} g_i z_i + \sum_{k\in K}\sum_{j\in J}(c_{kj} - \lambda_j)x_{kj}\right\} + \sum_{j\in J}\lambda_j$$

s.t.

$$y_i \geq t_k,\ \ k \in K_i,\ i \in V;$$

$$z_i \geq t_k,\ \ k \in K_i,\ i \in U;$$

$$t_k \geq x_{kj},\ \ k \in K, j \in J;$$

$$\sum_{j\in J} q_j x_{kj} \leq Q_k,\ \ k \in K;$$

$$x_{kj} \geq 0,\ \ k \in K, j \in J;$$

$$t_k, y_i, z_i \in \{0,1\}.$$

1. If $t_k = 0$, then $x_{kj} = 0$
2. If $t_k = 1$, then we have the knapsack problem with continues x_{kj}:

$$\min \sum_{j \in J} (c_{kj} - \lambda_j) x_{kj}$$

s.t.

$$\sum_{j \in J} q_j x_{kj} \le Q_k\ ;$$

$$0 \le x_{kj} \le 1.$$

Denote by $a_k = \sum_{j \in J} (c_{kj} - \lambda_j) x^*_{kj} \le 0$ the optimal value of the problem.

We can rewrite LR_1 as follower:

$$\min\left\{\sum_{i\in V} f_i y_i + \sum_{i\in U} g_i z_i + \sum_{k\in K} a_k t_k\right\}$$

s.t.

$$y_i \geq t_k,\ \ k \in K_i,\ i \in V;$$

$$z_i \geq t_k,\ \ k \in K_i,\ i \in U;$$

$$t_k, y_i, z_i \in \{0,1\}.$$

Can we solve this problem in polynomial time?

Is it possible to rewrite this problem as minimization problem for a pseudo Boolean function?

Let us remind the definition of the totally unimodular matrix

Definition. An $m \times n$ integer matrix A is *totally unimodular* (TU) if the determinant of the each square submatrix of A is equal to 0, 1, or -1.

If A is TU then $a_{ij} = 0, 1,$ or -1, but inverse is not true.

Theorem 5.3. If A is TU then each extreme point of polyhedral $P(b) = \{x \in R_+^n \mid ax \leq b\}$ is integral for all $b \in Z^m$ which it is not empty.

The decision problem: *Given A, is it TU?*

Does it belong to the class NP?

Theorem 5.4. If the $(0, 1, -1)$ matrix A has no more than two nonzero entries in each column, and if $\sum_i a_{ij} = 0$ if column j contains two nonzero coefficients, then A is TU (and the transpose of A is TU).

It is sufficient conditions for a matrix to be totally unimodular.

Theorem 5.5. The following statements are equivalent

1. A is TU.
2. For every $J \subseteq N = \{1, \dots, n\}$, there exists a partition J_1, J_2 of J such that

$$\left| \sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} \right| \leq 1$$

for $i = 1, \dots, m$.

Dynamic Facility Location Model

Given: T is the planning horizon;

f_{it} is the fixed cost of opening facility i in year t;

p_{it} is the production cost of facility i in year t;

c_{ijt} is the transportation cost for facility i to serve client j in year t;

q_{jt} is the demand of client j in year t;

Q_{it} is the capacity of facility i in year t.

Find: the minimal total cost for serving all clients during the planning horizon.

Mathematical Model

$$\min \sum_{t \in T} \sum_{i \in I} \left(f_{it} z_{it} + \sum_{j \in J} (c_{ijt} + q_{jt} p_{it}) x_{ijt} \right)$$

s.t.

$$\sum_{i \in I} x_{ijt} = 1, \quad j \in J,\ t \in T;$$

$$x_{ijt} \leq \sum_{\tau \leq t} z_{i\tau} \leq 1, \quad i \in I, j \in J, t \in T;$$

$$\sum_{j \in J} q_{jt} x_{ijt} \leq Q_{it} \sum_{\tau \leq t} z_{i\tau}, \quad i \in I, t \in T;$$

$$z_{it} \in \{0,1\}, \quad x_{ijt} \geq 0, \quad i \in I, j \in J, t \in T.$$

Lagrangian Heuristics

Let us relax the constraint

$$\sum_{i \in I} x_{ijt} = 1,\ j \in J, t \in T.$$

Include it with Lagrangian multipliers λ_{jt} into the objective fuction

$$+\sum_{j \in J} \sum_{t \in T} \lambda_{jt} \left(1 - \sum_{i \in I} x_{ijt} \right).$$

Can we solve the relaxation in polynomial time? $T = O(?),\ \Pi = O(?).$

Modify the model for the case when

1. some facilities are opened,
2. some facilities must be closed in nearest future,
3. capacities can enlarge by some additional payment,
4. we can buy some product by $\bar{p}_{it}$,
5. we have got some product at the beginning of planning,
6. rewrite the model to maximize the market share under budget constraints.

Hometask 2.

A producer of polyurethane has a stock of orders for the next 6 weeks. Let d_i be a parameter that denotes this known demand (say, in terms of number of gallons that must be delivered to customers during week i), and assume that $d_i > 0$ for all i. Let C_i denote the cost of producing a gallon during week i, and let K_i denote the maximum amount that can be produced (because of capacity limitations) in week i. Finally, let h_i denote the per unit cost of inventory in stock at the end of week i. (Thus, the inventory is measured as the number of gallons carried from week i into week $i + 1$). Suppose that the initial inventory (at the beginning of period 1 and for which no carrying charge is assessed) is known to be I_0 gallons. Find a production and inventory-holding plan that satisfies the known delivery schedule over the next 6 weeks at minimum total cost.

Formulate the constrained optimization model.