Lecture 7

Facility Location Games

Thus far we have assumed that there was only ***one decision maker*** who tried to minimize the total cost of opening facilities and servicing clients. However, clients may be free to choose the facility. They may have their ***own preferences***, for example, the travel time to a facility. They do not have to minimize the production and transportation costs of the firm. Hence, we should include client preferences in the mathematical model.

Let the matrix (g_{ij}) define the client preferences on the set I of potential facilities. If $g_{i_1 j} < g_{i_2 j}$, then client j prefers facility i_1. For simplicity we assume that all elements are different in each column of the matrix. Now we wish to find a subset of opening facilities in such a way that all clients will be serviced with minimal total cost, taken into account ***client preferences***.

Facility Location with Client Preferences

Given: I is the set of facilities;

J is the set of clients;

f_i is the fixed cost of opening facility i;

c_{ij} is the transportation cost of servicing client j from facility i;

g_{ij} is the client preferences.

Find: A subset of facilities to service all clients with minimal total cost taken into account the client preferences.

Mathematical Model (FLCP)

$$\min\left(\sum_{i\in I} f_i y_i + \sum_{j\in J}\sum_{i\in I} c_{ij} x^*_{ij}(y_i)\right)$$

s.t.

$$y_i \in \{0,1\}, \qquad x^*_{ij}(y_i) \in \mathcal{F}(y_i),$$

where $\mathcal{F}(y_i)$ is the set of optimal solutions of the client problem:

$$\min \sum_{j\in J}\sum_{i\in I} g_{ij} x_{ij}$$

s.t.

$$\sum_{i\in I} x_{ij} = 1, \;\; j \in J;$$

$$y_i \geq x_{ij}, \;\; i \in I, j \in J;$$

$$x_{ij} \in \{0,1\}, \;\; i \in I, j \in J.$$

Example. Greedy heuristic for the FLCP

$I = J = \{1, \dots, 6\}$

$$g_{ij} = \begin{pmatrix} 4 & 3 & 5 & 1 & 6 & 5 \\ 5 & 4 & 6 & 2 & 5 & 6 \\ 6 & 5 & 2 & 6 & 4 & 3 \\ 1 & 6 & 3 & 3 & 3 & 4 \\ 2 & 1 & 4 & 4 & 2 & 1 \\ 3 & 2 & 1 & 5 & 1 & 2 \end{pmatrix} \qquad c_{ij} = \begin{pmatrix} 0 & 169 & 291 & 88 & 153 & 208 \\ 169 & 0 & 248 & 75 & 112 & 199 \\ 291 & 248 & 0 & 231 & 146 & 132 \\ 88 & 75 & 231 & 0 & 93 & 137 \\ 153 & 112 & 146 & 93 & 0 & 88 \\ 208 & 199 & 132 & 137 & 88 & 0 \end{pmatrix} \qquad f_i = \begin{pmatrix} 150 \\ 150 \\ 150 \\ 100 \\ 100 \\ 100 \end{pmatrix}$$

Theorem 6.1. The FLCP problem is NP-hard in the strong sense even for $f_i = 0$ for all $i \in I$.

Proof. We reduce the SPLP to the FLCP problem. In the SPLP we have the set F for potential facilities and the set U of clients; r_i is the fixed cost of opening facility i and t_{ij} is the transportation cost. We put $I = F$, $J = U \cup \{1, \dots, |F|\}$, $f_i = 0$ for all $i \in I$ and

$$c_{ij} = \left(\begin{array}{c|cccc} & r_1 & & & 0 \\ & & r_2 & & \\ t_{ij} & & & \ddots & \\ & & & & \ddots \\ & 0 & & & r_{|F|} \end{array} \right) \qquad g_{ij} = \left(\begin{array}{c|cccc} & 1 & & & (i+j) \\ & & 1 & & \\ t_{ij} & & & \ddots & \\ & & & & \ddots \\ & (i+j) & & & 1 \end{array} \right)$$

For arbitrary solution $S \subseteq F$ of the SPLP we have the objective value

$$\sum_{i \in S} r_i + \sum_{j \in U} \min_{i \in S} t_{ij}.$$

For the same solution $S \subseteq F$ of the FLCP we have

$$\sum_{i \in S} f_i + \sum_{j \in J} c_{ij} x_{ij}^*(S) =$$

$$= \sum_{j \in U} \min_{i \in S} t_{ij} + \sum_{j=1}^{|F|} c_{ij} x_{ij}^*(S) =$$

$$= \sum_{j \in V} \min_{i \in S} t_{ij} + \sum_{i \in S} r_i \,.$$

Single Level Reformulation

Let the ranking for client j be $g_{i_1 j} < g_{i_2 j} < \cdots < g_{i_m j}$.

Put $S_{ij} = \{l \in I \,|g_{lj} < g_{ij}\}$ for all $i \in I, j \in J$.

If $x^*_{ij} = 1$, then $y_l = 0$ for all $l \in S_{ij}$. Therefore, we may rewrite FLCP as follows:

$$\min \sum_{i \in I} f_i y_i + \sum_{j \in J} \sum_{i \in I} c_{ij} x_{ij}$$

s.t.

$$x_{ij} + y_l \leq 1, \quad l \in S_{ij}, i \in I, j \in J;$$

$$\sum_{i \in I} x_{ij} = 1, \quad j \in J;$$

$$y_i \geq x_{ij}, \quad i \in I, j \in J;$$

$$y_i, x_{ij} \in \{0, 1\}, \quad i \in I, j \in J.$$

Others reformulations

$$\sum_{l \in S_{ij}} y_l \leq |S_{ij}|(1 - x_{ij}), \;\; i \in I, j \in J,$$

or

$$y_i \leq x_{ij} + \sum_{l \in S_{ij}} y_l, \;\; i \in I, j \in J,$$

or

$$y_i \leq x_{ij} + \sum_{l \in S_{ij}} x_{lj}, \;\; i \in I, j \in J.$$

Hometask 1. Show that the last inequality produces a better linear programming relaxation than the three previous ones.

Reduction to the Pseudo-Boolean Function

Using the ranking for client j $\quad g_{i_1 j} < g_{i_2 j} < \cdots < g_{i_m j}$

we put

$$\nabla c_{i_1 j} = c_{i_1 j};$$

$$\nabla c_{i_l j} = c_{i_l j} - c_{i_{l-1} j}, \;\; 1 < l < m,$$

Define the pseudo-boolean function $B(z)$ in the following way:

$$B(z) = \sum_{i \in I} f_i (1 - z_i) + \sum_{j \in J} \sum_{l \in I} \nabla c_{ij} \prod_{l \in S_{ij}} z_l.$$

Theorem 6.2. The FLCP problem is equivalent to the minimization problem for the pseudo-boolean function $B(z)$ for $z \neq (1, \ldots, 1)$. For the optimal solutions z^*, y^* of these problems we have $z_i^* = 1 - y_i^*$ for all $i \in I$.

Hometask 2. WSD Corporation is trying to complete its investment plans for the next two years. It has \$ 2 000 000 on hand and available for investment. In 6 months, 12 months, and 18 months, WSD expects to receive \$ 500 000, \$ 400 000, \$380 000 income respectively from previous investments.

There are two projects in which WSD is considering participation:

1. The Foster City Development
2. Middle Incoming Housing

The cash flow streams for the projects, at a 100% level of participation, would be as shown in Table:

	Initial	6 months	12 months	18 months	24 months
1 FCD	$\$ -1\,000\,000$	$\$ -700\,000$	$\$ \ 1\,800\,000$	$\$ \ 400\,000$	$\$ \ 600\,000$
2 MIH	$\$ -800\,000$	$\$ \ 500\,000$	$\$ -200\,000$	$\$ -700\,000$	$\$ \ 2\,000\,000$

Because of company policy, WSD is not permitted to borrow money. It can participate in any project at a level less than 100%, in which case all of the cash flows of that project are reduced proportionately. Management's goal is to maximize the cash on hand at the end of 24 months. Formulate this problem as LP model.