Mathematical Models in Logistics

Kochetov Yury Andreevich

NSU MMF 2015

http://www.math.nsc.ru/LBRT/k5/mml.html

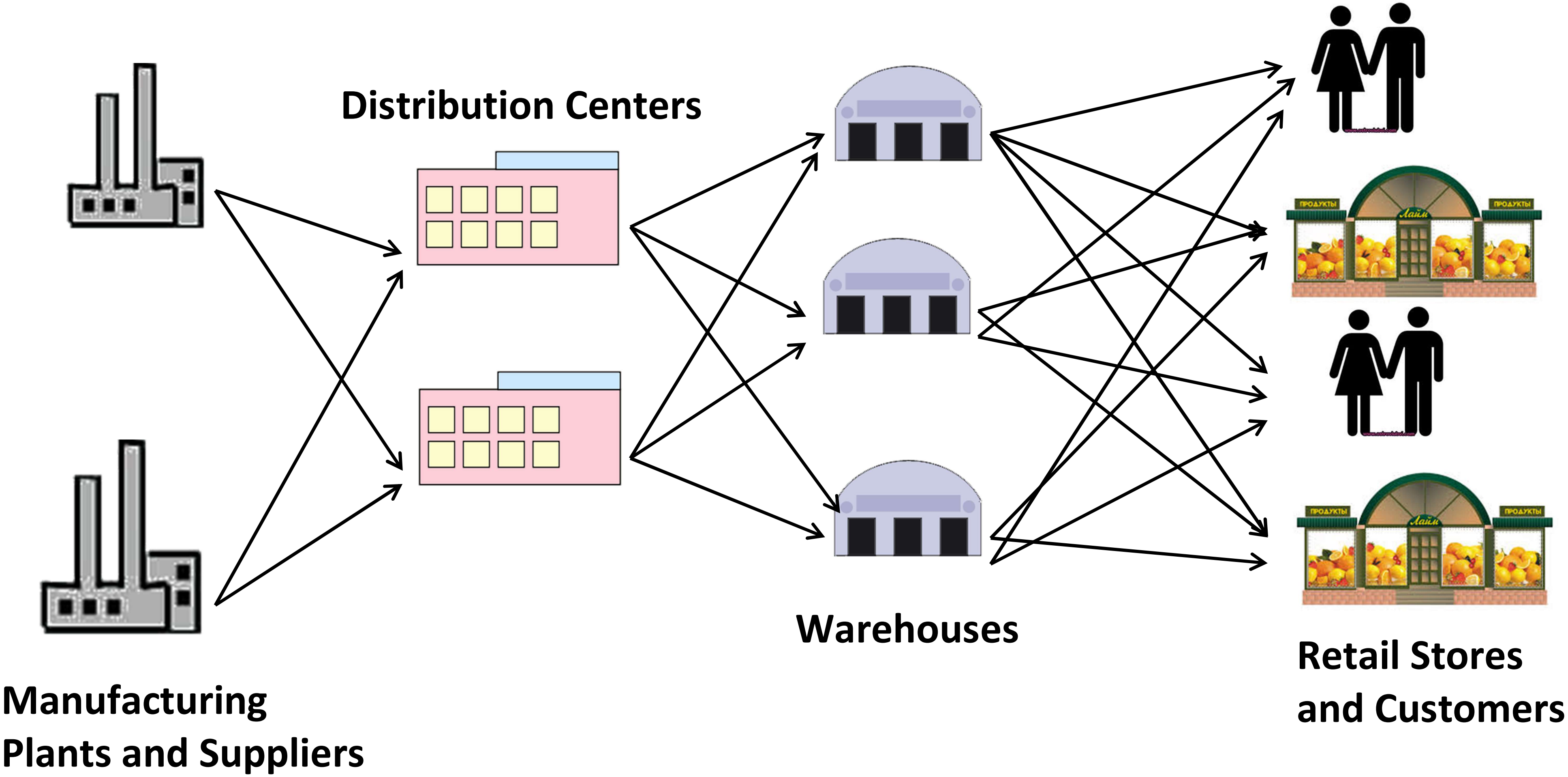
Science of Logistics Management

"Logistics management is the process of planning, implementing, and controlling the efficient, effective flow and storage of goods, services, and related information from point of origin to point of consumption for the purpose of conforming to customer requirements".

Council of Logistics Management

nonprofit organization of business personnel

The Logistics Network



Three Levels of Logistical Decisions

Because logistics management evolves around ***planning, implementing***, and ***controlling*** the logistics network, the decisions are typically classified into three levels:

- The **strategic level** deals with decisions that have a long-lasting effect on the firm. This includes decisions regarding the number, location and capacities of warehouses and manufacturing plants, or the flow of material through the logistics network.
- The **tactical level** typically includes decisions that are updated anywhere between once every week, month or once every quarter. This includes purchasing and production decisions, inventory policies and transportation strategies including the frequency with customers are visited.
- The **operational level** refers to day-to-day decisions such as scheduling, routing and loading trucks.

1. Network configuration (strategic level)

Several plants are producing products to serve a set of geographically dispersed retailers. The current set of facilities (plant and warehouses) is deemed to be inappropriate. We need to reorganize or redesign the distribution network: to change demand patterns or the termination of a leasing contract for a number of existing warehouses, change the plant production levels, to select new suppliers, to generate new flows of goods throughout the distribution network and others.

The goal is to choose a set of facility locations and capacities, to determine production levels for each product at each plant, to set transportation flows between facilities in such a way that total production, inventory and transportation costs are minimized and various service level requirements are satisfied.

2. Production Planning (strategic level)

A manufacturing facility must produce to meet demand for a product over a fixed finite horizon. All orders have been placed in advance and demand is known over the horizon.

Production costs consist of a fixed cost to machine set-up and a variable cost to produce one unit. A holding cost is incurred for each unit in inventory. The planer's objective is to satisfy demand for the product in each period and to minimize the total production and inventory costs over the fixed horizon.

The problem becomes more difficult as the number of products manufactured increases.

3. Inventory Control and Pricing Optimization (tactical level)

A retailer maintains an inventory of a particular product. Since customer demand is random, the retailer only has information regarding the probabilistic distribution of demand. The retailer's objective is to decide at what point to order a new batch of products, and how much to order. Ordering costs consists of two parts: fixed cost (to send a vehicle) and variable costs (the size of order). Inventory holding cost is incurred at a constant rate per unit of product per unit time. The price at which the product is sold to the end customer is also a decision variable. The retailer's objective is thus to find an inventory policy and a pricing strategy maximizing expected profit over the finite, or infinite, horizon.

4. Vehicle Fleet Management (operational level)

A warehouse supplies products to a set of retailers using a fleet of vehicles of limited capacity. A dispatcher is in charge of assigning loads to vehicles and determining vehicle routes. First, the dispatcher must decide how to partition the retailers into groups that can be feasibly served by a vehicle (loads fit in vehicle). Second, the dispatcher must decide what sequence to use so as to minimize cost.

One of two cost functions is considered:

- minimize the number of vehicles used;
- minimize the total distance traveled.

It is single depot capacitated vehicle routing problem: all vehicles are located in a warehouse (multiple depots, split delivery, open VRP, heterogeneous fleet, …).

5. Truck Routing

A truck leaves a warehouse to deliver products to a set of retailers. The order in which the retailers are visited will determine how long the delivery will take and what time the vehicle can return back to the warehouse. Each retailer can be visited in own time window. Moreover, the truck must spend a time for serving each retailer.

The problem is to find the minimal length route from a warehouse through a set of retailers.

It is an example of a traveling salesman problem with time windows.

6. Packing Problem

A collection of items must be packed into boxes, bins, or vehicles of limited size. The objective is to pack the items such that the number of bins used is as small as possible. This Bin-Packing Problem (BPP) appears as a special case of the CVRP when the objective is to minimize the number of vehicles used to deliver the products.

Cutting stock problems: 1,2,3 dimensional problems, rectangular items and figures, the knapsack problems for 2 dimensional items (load of vehicles), … .

Home task 1

Design a linear mixed integer programming model for the Truck Routing Problem:

Given: $J = \{0, \dots, n\}$ is the set of points in a city map, 0 is a warehouse, other point are retailers;

$c_{ij} \geq 0$ is the traveling time from point i to point j;

$p_i \geq 0$ is the serving time for retailer i;

$[e_i, l_i]$ is the time window for visiting retailer i. Truck can arrive to retailer i before e_i and waits.

Goal: Minimize the traveling time for serving all retailers. Truck starts own route from the warehouse and returns to it.

Designing the Logistics Network

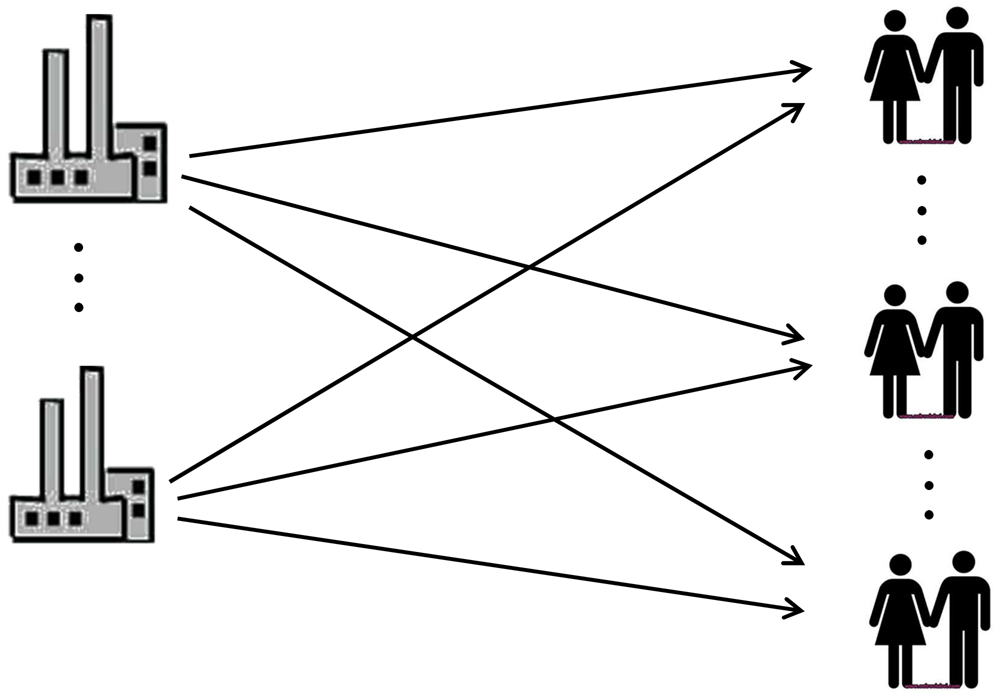
The network planning process **consists of** designing the system through which commodities flow from suppliers to demand points.

The main issues are to determine the number, location, equipment and size of new facilities.

The aim is the minimization of the annual total cost subject to constraints related to facility capacity and required customer service level.

Single–Echelon Single–Commodity Location Models

We are given a bipartite complete digraph $G(V_1 \cup V_2, A)$, where the vertices in V_1 stand for the potential facilities, the vertices in V_2 represent the customers, and the arcs in A are associated with material flows between the potential facilities and the demand points.



The SESC Model

Given:

$d_j, j \in V_2$ be the demand of customer j;

$q_i, i \in V_1$ be the capacity of potential facility i;

$u_i, i \in V_1$ be the production level of facility i; (*decision variable*)

$s_{ij}, i \in V_1, j \in V_2$ be the amound of product sent from i to j (*decision variable*)

$C_{ij}(s_{ij}), i \in V_1, j \in V_2$ be the cost of transporting s_{ij} units of product from i to j;

$F_i(u_i), i \in V_1$ be the cost for operating potential facility i at level u_i.

Goal:

Satisfy all customers demand and minimize the sum of the facility operating cost and transportation cost between facilities and customers.

Mathematical Model

$$\min \sum_{i \in V_1} \sum_{j \in V_2} C_{ij}(s_{ij}) + \sum_{i \in V_1} F_i(u_i)$$

subject to:

$$\sum_{j \in V_2} s_{ij} \le u_i, \quad i \in V_1;$$

$$\sum_{i \in V_1} s_{ij} = d_j, \quad j \in V_2;$$

$$u_i \le q_i, \quad i \in V_1;$$

$$s_{ij} \ge 0, \;\; u_i \ge 0, \qquad i \in V_1, j \in V_2.$$

Linear case

Let us assume that $C_{ij}(s_{ij}) = \bar{c}_{ij} s_{ij}$ and

$$F_i(u_i) = \begin{cases} 0 & \text{if } u_i = 0 \\ f_i + g_i u_i & \text{if } u_i > 0 \end{cases}$$

where f_i is a fixed cost, g_i is a marginal cost.

New variables:

$$y_i = \begin{cases} 1, & \text{if facility } i \text{ is opening} \\ 0 & \text{otherwise} \end{cases}$$

$x_{ij} \geq 0$ is the fraction of demand d_j satisfied by facility i.

We have:

$$s_{ij} = d_j x_{ij}$$

$$u_i = \sum_{j \in V_2} d_j x_{ij}$$

Mixed integer linear model

$$\min \sum_{i\in V_1}\sum_{j\in V_2} c_{ij}x_{ij} + \sum_{i\in V_1} f_i y_i$$

subject to

$$\sum_{i\in V_1} x_{ij} = 1, \quad j \in V_2;$$

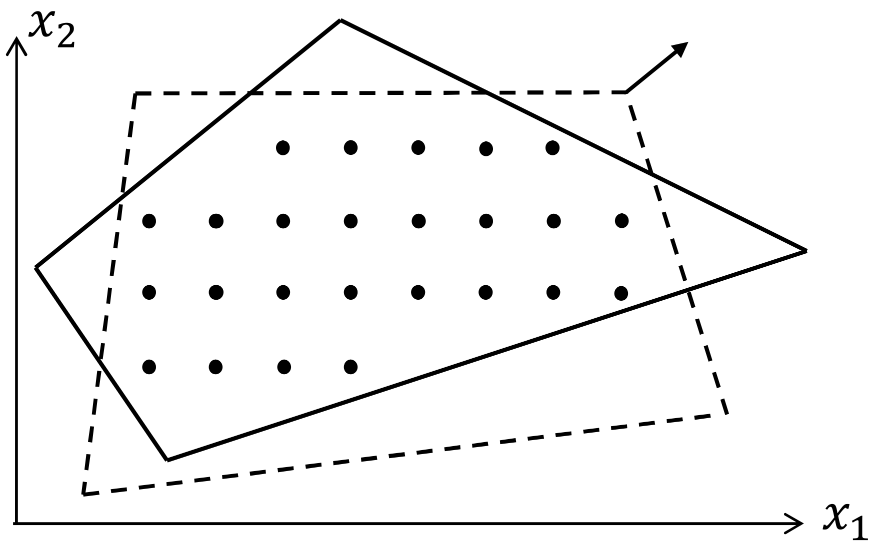
$$\sum_{j\in V_2} d_j x_{ij} \le q_i y_i, \quad i \in V_1;$$

$$0 \le x_{ij} \le 1, \quad y_i \in \{0,1\}, \qquad i \in V_1, j \in V_2;$$

where $$c_{ij} = d_j \bar{c}_{ij} + d_j g_i, \quad i \in V_1,\ j \in V_2.$$

Additional constraints: $x_{ij} \le y_i, \quad i \in V_1, j \in V_2.$ ***Is it useful?***

Comparison of reformulations



Integrality gap = $\frac{|\mathrm{Opt}_{LP} - \mathrm{Opt}_{IP}|}{\max\{\mathrm{Opt}_{IP}, \mathrm{Opt}_{LP}\}}$

What is the best reformulation?

How many new constraints and variables we have to use for it?

What is the cutting plane method?

The Simple Plant Location Problem

$$\min \sum_{i \in V_1} \sum_{j \in V_2} c_{ij} x_{ij} + \sum_{i \in V_1} f_i y_i$$

subject to

$$\sum_{i \in V_1} x_{ij} = 1, \quad j \in V_2;$$

$$x_{ij} \leq y_i, \quad i \in V_1, j \in V_2;$$

$$x_{ij} \in \{0,1\}, \quad y_i \in \{0,1\}, \qquad i \in V_1, j \in V_2.$$

Theorem 1. The SPLP is NP-hard.

We reduce the Node Cover Problem to the SPLP.

The NCP: given a graph G and an integer K, does there exist a subset of k nodes of G that cover all the arc of G? Node v covers arc e if v is an endpoint of e. The NCP is NP-complete.

Proof. Consider a graph $G = (V, E)$ with node set V and arc set E. We construct an instance of the SPLP with the set of potential facilities $V_1 = V$ and set of customers $V_2 = E$. Let $c_{ij} = 0$ if $v_i \in V_1$ is an endpoint of $e_j \in E$ and let $c_{ij} = 2$ otherwise. Also let $f_i = 1$ for all $i \in V_1$. This tramsformation is polynomial in the size of the graph.

An instance of the SPLP defined in this way consists of covering all the arcs of the graph G with the minimal number of nodes. Thus, an optimal solution of the SPLP provides the answer to the NCP. This proves that the SPLP is NP-hard.

MIT OpenCourseWare

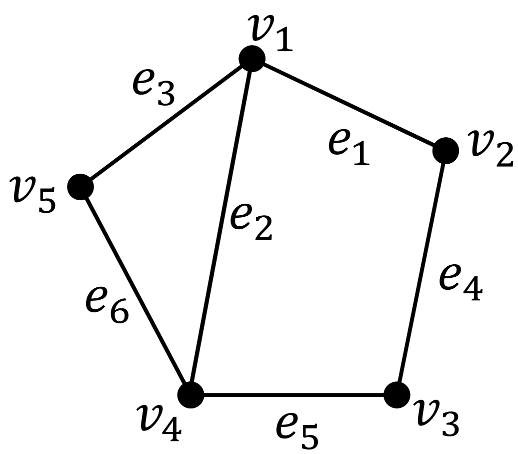
R23. Computational Complexity:

https://www.youtube.com/watch?v=t5Wxk96QjUk

Lecture 16: Complexity: P, NP, NP-completeness, Reductions

https://www.youtube.com/watch?v=eHZifpgyH_4

Example



$$c_{ij} = \begin{vmatrix} 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 0 & 0 & 2 \\ 2 & 0 & 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 2 & 2 & 0 \end{vmatrix}$$

In the example, $x_1 = x_3 = x_4 = 1,\ x_2 = x_5 = 0$ is an optimal solution to the instance. Hence, three nodes are needed to cover all of the arcs of G.

Can we assert that the SPLP is NP-hard in the strong sense? (Home task 2)

What can we say about the interality gap for the SPLP?

Theorem 2 *(J. Krarup P. Pruzan)*

The integrality gap for the SPLP can be arbitrary close to 1.

The Uniform Cost Model

The values c_{ij} are chosen independently and uniformly at random in some interval, say $[0,1]$.

Theorem 3 *(Ahn et al.)*

Assume that for some $\varepsilon > 0$ we have

$$n^{-(1/2)+\varepsilon} \leq f \leq n^{1-\varepsilon}$$

and $f_i = f$ for all $i \in V_1$. Then the integrality gap $\approx 0{,}5$ almost surely for the uniform cost model.

The Euclidean Cost Model

We choose n points $x_1, \dots, x_n$ independently and uniformly at random in the unit square $[0,1]^2$. Denote by $\|x_i = x_j\|$ the Euclidean distance between points x_i, x_j and let $f_i = f$ for all $i \in V_1$, $c_{ij} = \|x_i - x_j\|$, $i \in V_1, j \in V_2$ and $V_1 = V_2 = \{x_1, \dots, x_n\}$.

Theorem 4 *(Ahn et al.)*

Assume that for some $\varepsilon > 0$ we have

$$n^{-(1/2)+\varepsilon} \le f \le n^{1-\varepsilon}.$$

Then the integrality gap $\approx 0{,}00189 \dots$ almost surely for the Euclidean cost model.

Questions

1. Does there exist an exact polynomial time algorithm for the SPLP ?
2. The Single-Eshelon Single-Commodity Location Problem is NP-hard. (?)
3. The Truck Routing Problem is NP-hard. (?)
4. If the integrality gap is 0 then the problem is polynomially solvable. (?)
5. For the combinatorial optimization problems can exist several equivalent reformulations. (?)