On Cavicchioli-Hegenbarth-Repovš groups

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Abstract

The class of cyclically presented groups which contains Fibonacci groups and Sieradski groups is studied. There are obtained conditions for these groups to be finite, pairwise isomorphic, or aspherical. As a partial answer on the question of Cavicchioli, Hegenbarth and Repovš, it is shown that the wide subclass of groups with odd number of generators can not be realized as fundamental groups of hyperbolic three-dimensional manifolds of finite volume.

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1. Introduction

In the present paper the class of cyclically presented groups introduced by Cavicchioli, Hegenbarth and Repovš in [1] is studied. The interest to this class is motivated by the following. From one side, this class contains such well-known groups as the Fibonacci groups and the Sieradski groups. From the other side, the following question was posed in [1]: whether groups from this class are fundamental groups of three-dimensional manifolds?

The fundamental groups of compact 2-dimensional manifolds are well-studied [2]. At the same time [3, § 5.1], for $n \geq 4$ any finitely defined group can be realized as fundamental group of some closed orientable n-dimensional manifold. The case of 3-dimensional manifolds is much more complicated. It was shown by Stallings [4], that there no algorithm which for a given finite presentation of a group admits to say whether this group is fundamental group of a 3-dimensional manifold.

The problem of distinguishing of groups of 3-manifolds is important in 3-dimensional topology, because fundamental group is one of the most valuable invariants of a manifold. This problem is also important in group theory, because the knowledge that the group is fundamental group of a 3-manifold can gives us more information about the structure of the group. In particular, if G is fundamental group of a 3-manifold of constant negative curvature, then it is hyperbolic in the sense of Gromov, and therefore, the equality problem and conjugacity problem are solvable in G.

One of classes of groups for which the problem of distinguishing of 3-manifold groups seems most actual, is the class of cyclically presented groups. We say that a group G is *cyclically presented* if it has the following presentation:

$$G_n(w) = \langle x_1, x_2, \ldots, x_n \mid w = 1, \eta(w) = 1, \ldots, \eta^{n-1}(w) = 1 \rangle,$$

where w is a word in the alphabet $X = \{x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}\}$, and η is the automorphism of the free group $F_n = F_n(x_1, \ldots, x_n)$ defined by its action on the generators by the following rule: $\eta(x_i) = x_{i+1}$ for $i = 1, \ldots, n$, where all indices are taken by mod n.

Algebraic properties of cyclically presented groups (their finiteness and abelian invariants) were studied in [5, 6, 7, 8, 9]. Three-dimensional manifolds fundamental groups of which are cyclically presented were investigated in [10, 11, 12, 15, 13, 14, 16, 17]. In particular, there was especially studied the case when the automorphism of $G_n(w)$, induced by the above automorphism η of F_n, corresponds to the cyclic branched covering of the 3-dimensional sphere (see [18, 19, 20, 21, 22, 23]).

In the present paper there are considered cyclically presented groups with the defining word $w = x_1 x_{1+m} x_{1+k}^{-1}$ for some integers m and k, i.e. groups

$$G_n(m, k) = G_n(x_1 x_{1+m} x_{1+k}^{-1}) = \langle x_1, x_2, \ldots, x_n \mid x_i x_{i+m} = x_{i+k},\ i = 1, \ldots, n \rangle,$$

where all indices are taken by mod n and belong to the set $\{1, 2, \ldots, n\}$.

Remark that the class of groups $G_n(m, k)$ contains some well-known groups studied by many authors before. For $m = 1$ and $k = 2$ we have $G_n(1, 2) \cong F(2, n)$, the Fibonacci groups introduced by Conway in [5]. It was shown in [21] by Helling, Kim and Mennicke that for each even $n \geq 4$ the group $F(2, n)$ is fundamental group of a 3-dimensional manifold which can be presented as the $(n/2)$-fold cyclic branched covering of the 3-sphere, branched over the figure-eight knot. Moreover, for $n \geq 8$ these manifolds are hyperbolic. ¿From the other side, it was pointed out by Maclachlan in [24] that for odd n the group $F(2, n)$ can not be fundamental group of a hyperbolic 3-orbifold (in particular, 3-manifold) of finite volume. Various generalizations of Fibonacci groups were considered in [7, 15, 16, 9]. Asphericity and atoricity of a wide class of generalized Fibonacci groups were investigated by Prischepov [25]. For $m = 2$ and $k = 1$ we have $G_n(2, 1) \cong S(n)$, the Sieradski groups investigated in [26], where it was shown that these groups are 3-manifold groups. Cavicchioli, Hegenbarth and Kim [27] proved that these manifolds can be obtained as the n-fold cyclic branched coverings of the 3-sphere, branched over the trefoil knot (see also [12] and [28]). For $k = 1$ we have the family of groups $G_n(m, 1)$ investigated by Gilbert and Howie [29]. They studied HNN–extensions of these group, their finiteness and asphericity. It was shown in [30], that the abelizator of $G_n(m, 1)$ is infinite if and only if n is divided by 6 and $m \equiv 2 \pmod 6$.

The present paper is organized as follows. In Section 2 some results about structure and pairwise isomorphisms of groups $G_n(m, k)$ are presented. There are obtained conditions for these groups to be cyclic, decomposable in a free products, and to be pairwise isomorphic. In Section 3 the approach from [29] is generalized, and using results from [31], the condition of asphericity for these groups is given. In Section 4 the partial answer on the question from [1] is obtained. More precisely, it is shown that a sufficiently wide subclass of groups $G_n(m, k)$ with odd number of generators can not be realized as fundamental groups of hyperbolic 3-orbifolds (in particular, hyperbolic 3-manifolds) of finite volume. In Section 5 there are listed orders of groups $G_n(m, k)$ and their abelizators for small values of parameters, obtained by using the computer program *GAP* [32]. In Section 6 some open problems about properties of groups $G_n(m, k)$ are formulated.

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2. Elementary properties of groups $G_n(m, k)$

Directly from the presentation of the group $G_n(m, k)$ one get

LEMMA 2.1. *Groups $G_n(m, k)$ have the following properties:*
1) If n and k are coprime then $G_n(0, k) \cong \mathbf{Z}_{2^n - 1}$.
2) If $k = 0$ or $m = k$ then $G_n(m, k)$ is trivial.
3) $G_n(m, k) \cong G_n(n - m, n - m + k)$.
4) $G_{2k}(k - 1, k) \cong G_{2k}(k + 1, 1) \cong \mathbf{Z}_{2^k + 1}$.

PROOF. 1) The group $G_n(0, k)$ has the following presentation:

$$G_n(0, k) = \langle x_1, \ldots, x_n \mid x_i^2 = x_{i+k}, \quad i = 1, 2, \ldots, n \rangle.$$

If $i = n - k$ we get the relation $x_{n-k}^2 = x_n$. Substituting this expression for x_n into the relation $x_n^2 = x_k$, we will get

$$G_n(0, k) = \langle x_1, \ldots, x_{n-1} \mid x_{n-k}^4 = x_k,\, x_i^2 = x_{i+k}, \quad i \neq n - k,\, i = 1, \ldots, n - 1 \rangle.$$

Analogously, we can eliminate the generator x_{n-1}, after that x_{n-2}, and so on. At the end of this process we will get

$$G_n(0,k) = \langle x_1 \mid x_1^{2^n} = x_1 \rangle \cong \mathbf{Z}_{2^n-1}.$$

2) The statement follows immediately from the definition of $G_n(m,k)$.

3) Denote $y_i = x_i^{-1}$ for $i = 1,2,\ldots,n$ and remark that $G_n(m,k)$ can be defined by the following presentation:

$$G_n(m,k) = \langle y_1,\ldots,y_n \mid y_{i+m}y_i = y_{i+k}, \quad i = 1,\ldots,n\rangle.$$

Setting $j = i+m$ we rewrite this system of defining relations in the form: $y_j y_{j-m} = y_{j-m+k}$, where $j = 1,\ldots,n$. Since all indices are by mod n, we see that

$$G_n(m,k) = \langle y_1,\ldots,y_n \mid y_j y_{j+(n-m)} = y_{j+(n-m+k)}, \quad i = 1,\ldots,n\rangle \; = G_n(n-m, n-m+k)\,.$$

4) The first isomorphism follows from item 3), and the second is established in [29, Proposition 2.2].

The proof of Lemma is completed.

In some cases $G_n(m,k)$ decomposes in a free product. Here and below we denote by $\gcd(a_1,a_2,\ldots,a_n)$ the greater common divisor of integers $a_1,a_2,\ldots,a_n$.

LEMMA 2.2. *For a given group $G_n(m,k)$ denote $u = \gcd(n,k)$ and $r = \gcd(n,k-m)$. Then*

1) For any positive integer ℓ the group $G_{\ell n}(\ell m,\ell k)$ is isomorphic to a free product of ℓ copies of $G_n(m,k)$.

2) If $\gcd(u,r) > 1$ then $\gcd(n,m,k) > 1$ and $G_n(m,k)$ decomposes into a non-trivial free product.

PROOF. 1) It is easy to see that for each $j = 1,\ldots,\ell$ the subgroup $G_j = \langle x_j, x_{j+\ell},\ldots,x_{j+\ell(n-1)}\rangle$ of $G_{\ell n}(\ell m,\ell k)$ is isomorphic to $G_n(m,k)$ and the above sets of generators of groups G_j and $G_{j'}$ do not intersect if $j \neq j'$. From the presentation of $G_{\ell n}(\ell m,\ell k)$ we see that it is isomorphic to the following free product: $G_1 * G_2 * \ldots * G_\ell$.

2) In this case n, k and m have a common divisor $d = \gcd(u,r) > 1$, and the statement follows from the previous item. The Lemma is proven.

Groups $G_n(t,1)$ were investigated in [29]. The following statement shows that in many cases groups $G_n(m,k)$ reduce to them.

LEMMA 2.3. *If $\gcd(n,k) = 1$ or $\gcd(n,m-k) = 1$ then $G_n(m,k)$ is isomorphic to $G_n(t,1)$ for some positive integer t.*

PROOF. Let $\gcd(n,k) = 1$. We reorder generators of $G_n(m,k)$ defining

$$c_1 = x_1,\; c_2 = x_{1+k},\; \ldots,\; c_i = x_{1+(i-1)k},\; \ldots,\; c_n = x_{1+(n-1)k}.$$

Remark that the set of generators $\{c_1,\ldots,c_n\}$ coincides with the set $\{x_1,\ldots,x_n\}$. Then the first relation $x_1 x_{1+m} = x_{1+k}$ will be rewritten in the form $c_1 c_{1+t} = c_2$, where $c_{1+t} = x_{1+m} = x_{1+tk}$ and t is defined from the condition $tk \equiv m \pmod{n}$. Next relation $c_2 c_{2+t} = c_3$ corresponds to the relation $x_{1+k}x_{1+k+m} = x_{1+2k}$, because $c_{2+t} = x_{1+(1+t)k} = x_{1+k+m}$. Analogously, $c_j c_{j+t} = c_{j+1}$ corresponds to the relation $x_{1+(j-1)k}\; x_{1+(j+t-1)k} = x_{1+jk}$ which is equivalent to $x_{1+(j-1)k}\; x_{1+(j-1)k+m} = x_{1+(j-1)k+k}$.

If j runs over the set $\{1,\ldots,n\}$ then $1+(j-1)k$, taken by mod n, runs over the set $\{1,\ldots,n\}$ too. Therefore, $G_n(m,k) \cong G_n(t,1)$.

The case $\gcd(n,m-k) = 1$, by item (iii) of Lemma 2.1, is equivalent to the above considered case. The Lemma is proven.

In virtue Lemma 2.1 and Lemma 2.2, we can consider below only groups $G_n(m,k)$ with parameters n, m and k satisfying the following conditions:

$$0 < m < k < n, \quad \gcd(n,m,k) = 1. \tag{1}$$

The group $G_n(m,k)$ with parameters satisfying conditions (1) will be referred as *irreducible* since otherwise it is trivial, or cyclic, or can be decomposed into a free product. Nevertheless, there are pairwise isomorphic groups among irreducible. More precisely, the following statement holds.

THEOREM 2.1. *Let $G_n(m,k)$ and $G_n(m',k')$ be two irreducible groups. If k' is divided by $r = \gcd(n, k-m)$ and if there exist positive integers i and j such that*

$$\begin{cases} i + j(k-m) \equiv 1 - m \pmod{n}, \\ m' + 1 \equiv i + jk' \pmod{n}, \\ 1 \leq i \leq r, \quad 1 \leq j \leq n/r, \end{cases} \tag{2}$$

then $G_n(m,k)$ is isomorphic to $G_n(m',k')$.

Before proving the theorem we will illustrate its application on two examples.

EXAMPLE 2.1. Let $n=4$, $m=m'=1$, $k=2$, $k'=3$. Then $r=1$ and numbers $i=1$ and $j=3$ satisfy system (2). It follows from Theorem 2.1 that $G_4(1,2) \cong G_4(1,3)$. Indeed, it is easy to check directly that $G_4(1,2) \cong G_4(1,3) \cong Z_5$.

EXAMPLE 2.2. Let $n=5$, $m=1$, $k=2$, $m'=2$, $k'=3$. Then $r=1$ and numbers $i=1$ and $j=4$ satisfy system (2). Therefore, by Theorem 2.1, $G_5(1,2) \cong G_5(2,3)$.

PROOF of the theorem. Consider an irreducible group

$$G_n(m,k) = \langle x_1, \ldots, x_n \mid x_i x_{i+m} = x_{i+k}, \quad i = 1, \ldots, n \rangle.$$

Defining $c_i = x_i^{-1}$, $i = 1, \ldots, n$, we will get:

$$G_n(m,k) = \langle c_1, \ldots, c_n \mid c_{i+m} c_i = c_{i+k}, \quad i = 1, \ldots, n \rangle.$$

Denote $j = i + m$. Then $i = j - m$ and the system of defining relations can be rewritten as following:

$$c_j c_{j-m} = c_{j-m+k}, \quad j = 1, \ldots, n.$$

Denote $\ell = n/r$, where $r = \gcd(n, k-m)$ and separate generators $c_1, \ldots, c_n$ into r groups with ℓ elements in each group:

$$\begin{aligned} A_1 &= \{c_1, \ c_{1+(k-m)}, \ c_{1+2(k-m)}, \ldots, \ c_{1+(\ell-1)(k-m)}\}, \\ A_2 &= \{c_2, \ c_{2+(k-m)}, \ c_{2+2(k-m)}, \ldots, \ c_{2+(\ell-1)(k-m)}\}, \\ &\cdots \\ A_r &= \{c_r, \ c_{r+(k-m)}, \ c_{r+2(k-m)}, \ldots, \ c_{r+(\ell-1)(k-m)}\}. \end{aligned}$$

Remark that the separation of generators in classes $A_1, \ldots, A_r$ induces the separation of relations in r classes $R_1, \ldots, R_r$ with ℓ relations in each of them:

$$\begin{aligned} R_1: \ & c_1 c_{1-m} = c_{1+(k-m)}, \ c_{1+(k-m)} c_{1+k-2m} = c_{1+2(k-m)}, \ldots, c_{1+(\ell-1)(k-m)} c_{1+(\ell-1)(k-m)-m} = c_{1+\ell(k-m)}. \\ R_2: \ & c_2 c_{2-m} = c_{2+(k-m)}, \ c_{2+(k-m)} c_{2+k-2m} = c_{2+2(k-m)}, \ldots, c_{2+(\ell-1)(k-m)} c_{2+(\ell-1)(k-m)-m} = c_{2+\ell(k-m)}. \\ & \cdots \\ R_r: \ & c_r c_{r-m} = c_{r+(k-m)}, \ c_{r+(k-m)} c_{r+k-2m} = c_{r+2(k-m)}, \ldots, c_{r+(\ell-1)(k-m)} c_{r+(\ell-1)(k-m)-m} = c_{r+\ell(k-m)}. \end{aligned}$$

Remark that $c_{p+\ell(k-m)} = c_p$ for $p = 1, \dots, r$. Indeed, since $r = \gcd(n, k-m)$, we suppose $n = rn_1$ and $k - m = rr_1$ where $n_1, r_1 \in \mathbf{N}$ and $\gcd(n_1, r_1) = 1$. Then $\ell(k-m) = (n/r)(k-m) = (n/r)rr_1 = nr_1 \equiv 0 \pmod{n}$ and hence $c_{p+\ell(k-m)} = c_p$. Therefore, for each relation from R_p the first generator from the left part and the generator from the right part are from the class A_p.

Consider the presentation of the second group:

$$G_n(m', k') = \langle y_1, \dots, y_n \mid y_i y_{i+m'} = y_{i+k'}, \quad i = 1, \dots, n \rangle.$$

Separate its generators in r classes with ℓ elements in each of them:

$$\begin{aligned} B_1 &= \{y_1,\ y_{1+k'},\ y_{1+2k'}, \dots,\ y_{1+(\ell-1)k'}\}, \\ B_2 &= \{y_2,\ y_{2+k'},\ y_{2+2k'}, \dots,\ y_{2+(\ell-1)k'}\}, \\ & \cdots \\ B_r &= \{y_r,\ y_{r+k'},\ y_{r+2k'}, \dots,\ y_{r+(\ell-1)k'}\}. \end{aligned}$$

Analogously to the previous case, let us separate defining relations of $G_n(m', k')$ in r classes with ℓ relations in each:

$$\begin{aligned} Q_1 :&\quad y_1 y_{1+m'} = y_{1+k'},\ y_{1+k'} y_{1+k'+m'} = y_{1+2k'}, \dots, y_{1+(\ell-1)k'} y_{1+(\ell-1)k'+m'} = y_{1+\ell k'}. \\ Q_2 :&\quad y_2 y_{2+m'} = y_{2+k'},\ y_{2+k'} y_{2+k'+m'} = y_{2+2k'}, \dots, y_{2+(\ell-1)k'} y_{2+(\ell-1)k'+m'} = y_{2+\ell k'}. \\ & \cdots \\ Q_r :&\quad y_r y_{r+m'} = y_{r+k'},\ y_{r+k'} y_{r+k'+m'} = y_{r+2k'}, \dots, y_{r+(\ell-1)k'} y_{r+(\ell-1)k'+m'} = y_{r+\ell k'}. \end{aligned}$$

Since $\ell k' = (n/r)k' \equiv 0 \pmod{n}$, using that r divides k', we get $y_{p+lk'} = y_p$, $p = 1, \dots, r$. Therefore, for each defining relation from Q_p the first generator from left part and the generator from right part are from the class B_p.

Define the correspondence $\varphi : G_n(m, k) \to G_n(m', k')$ by its action on generators:

$$\varphi(c_{p+q(k-m)}) = y_{p+qk'}, \quad 1 \le p \le r,\ 0 \le q \le \ell - 1,$$

and check that each defining relation of $G_n(m, k)$ goes to the defining relation of $G_n(m', k')$.

Indeed, consider the relation $c_1 c_{1-m} = c_{1+k-m}$. By the assumption of the theorem, there exist positive integers i and j with $1 \le i \le r$, and $1 \le j \le (n/r) = l$ such that $i + j(k-m) \equiv 1 - m \pmod{n}$. Therefore, the relation under consideration can be rewritten in the form $c_1 c_{i+j(k-m)} = c_{1+k-m}$. The image of this relation under φ is $y_1 y_{i+jk'} = y_{1+k'}$. Using the assumption $m' + 1 \equiv i + jk' \pmod{n}$ we will get the relation $y_1 y_{1+m'} = y_{1+k'}$ that is the defining relation in the irreducible group $G_n(m', k')$.

To complete the proof it is enough to remark that all defining relations of $G_n(m, k)$, as well as all defining relations of $G_n(m', k')$, arise from the first relation under the cyclic permutation of indices of generators in the first relation. Therefore, φ is a homomorphism. It is easy to see that it is convertible, and so, it is an isomorphism. Thus, the theorem is proven.

As a particular case of Theorem 2.1 we get the following result of Gilbert and Howie [29, Lemma 2.1].

COROLLARY 2.1. *Let n and t be such positive integers that $n > t$ and $\gcd(n, t-1) = 1$. Let s be such integers that $0 \le s < n$ and $t \equiv (t-1)s \pmod{n}$. Then groups $G_n(t, 1)$ and $G_n(s, 1)$ are isomorphic.*

PROOF. By item 3) of Lemma 2.1 we have $G_n(t, 1) \cong G_n(n-t, n-t+1)$ and $G_n(s, 1) \cong G_n(n-s, n-s+1)$. Thus, it is enough to proof that groups $G_n(n-t, n-t+1)$ and $G_n(n-s, n-s+1)$ are isomorphic. Setting $m = n - t$, $k = n - t + 1$, $m' = n - s$ and $k' = n - s + 1$, we see that

$r = \gcd(n, k-m) = 1$, and the assumption of Theorem 2.1 that k' is divided by r holds. Considering $i = 1$ and $j = t$ in the system of conditions (2), we will get:

$$\begin{cases} 1 + t \equiv 1 + t + n \pmod{n}, \\ n - s \equiv 1 + t(n - s + 1) - 1 \pmod{n}. \end{cases}$$

The first condition holds obviously. The second condition is equivalent to $t \equiv s(t-1) \pmod{n}$ that holds by the assumption. The Corollary is proven.

LEMMA 2.4. *Let n, m and ℓ such positive integers that $\gcd(n, m) = 1$, $\ell \geq 2$, and $\ell m < n$. Then*

1) $G_n(m, \ell m)$ is isomorphic to $G_n(1, \ell)$;

2) $G_n(\ell m, m)$ is isomorphic to $G_n(\ell, 1)$.

PROOF. 1) Consider groups

$$G = G_n(1, \ell) = \langle x_1, x_2, \ldots, x_n \mid x_i x_{i+1} = x_{i+\ell},\ i = 1, \ldots, n \rangle$$

and

$$H = G_n(m, \ell m) = \langle y_1, y_2, \ldots, y_n \mid y_i y_{i+m} = y_{i+\ell m},\ i = 1, \ldots, n \rangle.$$

Define the correspondence $\varphi : G \to H$ by its action on the generators:

$$\varphi(x_j) = y_{1+m(j-1)}, \qquad j = 1, 2, \ldots, n.$$

Obviously this is a map "onto". Let us check that φ maps defining relations of G to defining relations of H. Indeed,

$$\varphi\left(x_i\, x_{i+1}\, x_{i+\ell}^{-1}\right) = y_{1+m(i-1)}\, y_{1+mi}\, y_{1+m(i+\ell-1)}^{-1} = y_j\, y_{j+m}\, y_{j+\ell m}^{-1},$$

where $j = 1 + m(i-1)$. Therefore, φ is a homomorphism. Since φ is convertible, it is an isomorphism.

The statement 2) follows by similar considerations. The Lemma is proven.

The above remarked property admits to reduce studding of some groups to the well-known Fibonacci groups and Sieradski groups.

COROLLARY 2.2. *1) A group $G_n(m, 2m)$ either is isomorphic to the Fibonacci group $F(2, n) \cong G_n(1, 2)$, or is a free product.*

2) A group $G_n(2m, m)$ either is isomorphic to the Sieradski group $S(n) \cong G_n(2, 1)$, or is a free product.

PROOF. 1) If $\gcd(n, m) \neq 1$ then by item (ii) of Lemma 2.2 the group is a free product. If $\gcd(n, m) = 1$ then by Lemma 2.1 we can suppose that $2m < n$. Then, by Lemma 2.4 $G_n(m, 2m)$ is isomorphic to $G_n(1, 2)$.

The proof of the statement 2) is analogously. The Corollary is proven.

3. Asphericity of groups $G_n(m, k)$

In the investigation of topological properties of groups the question about asphericity of their presentations naturally arises (see [33, 34]). Roughly speaking, asphericity of a presentation means that there are no nontrivial identities between defining relations.

Asphericity of groups $G_n(t,1)$ was studied by Gilbert and Howie in [29], where they gave conditions of asphericity for values of parameters (n,t) different from $(8,3)$, $(9,4)$ and $(9,7)$. Remark that groups $G_n(m,k)$ are particular cases of the following groups introduced by Prischepov in [25]:

$$P(r,n,k,s,q) = \langle x_1,\ldots,x_n \mid x_i x_{i+q}\cdots x_{i+q(r-1)} = x_{i+k}x_{i+k+q}\cdots x_{i+k+q(s-1)}\rangle$$

where $i=1,\ldots,n$ and all indices are taken by mod n. Obviously, we have $G_n(m,k)\cong P(2,n,k,1,m)$. In [25] there were obtained conditions of asphericity and atoricity for groups $P(r,n,k,s,q)$ with the assumption $r>2s>0$. The results obtained in this section corresponds to the case $r=2s$ with $s=1$, and in this sense are complementary to results from [25]. Asphericity of generalized Fibonacci group $F(r,2r+1)$, $F(r,2r)$ and some others, not considered in [25], was investigated in [35].

In this section we will obtain the condition of asphericity for presentations $G_n(m,k)$, as a generalization of results from [29].

Following [33], we recall some facts about asphericity of group and asphericity of relative presentations, which will be used below.

An *relative presentation* is a triple $\mathbf{P}=\langle H,X|R\rangle$, where H is a group, $X=\{x_1,x_2,\ldots\}$ is a set, and R is a set of words in the alphabet $H\cup X\cup X^{-1}$ of the form $r=y_1h_1y_2h_2\cdots y_nh_n$, where $y_i\in X\cup X^{-1}$, $h_i\in H$. We suppose that words are cyclically reduced in the following sense: if $h_i=1$ then $y_{i+1}\neq y_i^{-1}$, where all indices by mod n. The elements of $X\cup X^{-1}$ will be called X-*symbols*.

The words in R represent elements of the free product $H*\langle X\rangle$. The presentation $\mathbf{P}$ defines a group which is the quotient group of the group $H*\langle X\rangle$ by the normal closer of the set R.

For a given subset $S\subseteq R$ define by S^* the set of all cyclic permutations of words from $S\cup S^{-1}$ which begins from X-symbols.

Let us define the following operator on R^*. Presenting a word $r\in R^*$ in the form $r=sh$, where $h\in H$ and s begins and ends with X-symbols, we define $\overline{r}=s^{-1}h^{-1}$. Remark that $\overline{\overline{r}}=r$ and $\overline{r}\in R^*$.

A *picture* $\mathcal{P}$ is a closed disc D^2 whose interior contains a finite collection $\{\Delta_1,\ldots,\Delta_d\}$ of pairwise disjoint closed discs together with a finite collection $\{\alpha_1,\ldots,\alpha_q\}$ of pairwise disjoint compact connesced 1-dimensional manifolds α_j, $j=1,\ldots,q$ called *arcs*, such that the union of their boundaries belong to boundaries of discs: $\cup_{j=1}^q\partial\alpha_j\subset\partial D^2\cup\cup_{i=1}^d\partial\Delta_i$. The *boundary* $\partial\mathcal{P}$ of the picture $\mathcal{P}$ is the boundary ∂D^2 of D^2. *Corners* of a disc Δ_i, $i=1,\ldots,d$ are closes of the connected components of $\partial\Delta_i\setminus\cup_{j=1}^q\alpha_j$, where $\partial\Delta_i$ is the boundary of the disc Δ_i. The *regions* of $\mathcal{P}$ are the closures of the connected components of $D^2\setminus\left(\cup_{i=1}^d\Delta_i\cup\cup_{j=1}^q\alpha_j\right)$. A simply-connected region is said to be *interior* region of $\mathcal{P}$ if it does not meet $\partial\mathcal{P}$. The picture $\mathcal{P}$ is *non-trivial* if $d\geq 1$; is *connected* if $\cup_{i=1}^d\Delta_i\cup\cup_{j=1}^q\alpha_j$ is connected; and is *spherical* if it is non-trivial and $\cup_{j=1}^q\alpha_j\cap\partial\mathcal{P}=\emptyset$.

For a given relative presentation $\mathbf{P}=\langle H,X|R\rangle$ a picture $\mathcal{P}$ is said to be *labeled* if the following conditions are satisfied:
(i) each arc is to be equipped with a normal orientation, indicated by a short arrow meeting the arc transversely and labeled by an element of $X\cup X^{-1}$;
(ii) each corner of the picture $\mathcal{P}$ is to be oriented anticlockwise and labeled by an element of H.

For an corner c of a disc Δ of a labeled picture $\mathcal{P}$ we denote by $\omega(c)$ the word that arises when we reading in anticlockwise order the labels on the arcs and corners meeting $\partial\Delta$ beginning with the label on the arc following after the corner c. In this case, a label t gives us t if the orientation of the arc coincides with the orientation in which we read, and t^{-1} otherwise.

A labeled picture $\mathcal{P}$ is said to be a *picture over the relative presentation* $\mathbf{P}$ if the following conditions are satisfied:
(1) if c is an corner of $\mathcal{P}$ then $\omega(c)\in R^*$;
(2) if $h_1,h_2,\ldots,h_m$ is the sequence of all labels of corners listed in the anticlockwise direction along boundary of some interior disc, then $h_1h_2\cdots h_m=1$ in H.

An ordinary presentation $\mathbf{Q} = \langle X|R\rangle$ can be considered as the particular case of a relative presentation for which $H = \{1\}$. In this case every corner of a picture have labeled by 1 and the condition (2) is satisfied. Ignoring these 1's labels we get the (ordinary) picture over the (ordinary) presentation $\mathbf{Q}$ (see [33]).

Let us return to a relative presentation $\mathbf{P}$. A *dipole* in a picture over $\mathbf{P}$ is a pair of corners c and c' with an arc α connecting the beginning of one corner with the end of other corner in such a way that c and c' belong to the same region of the picture and $\omega(c') = \overline{\omega(c)}$.

A picture over $\mathbf{P}$ is said to be *reduced* if it doesn't contain dipoles. A relative presentation $\mathbf{P}$ is said to be *aspherical* if every connected spherical picture over $\mathbf{P}$ contains a dipole, i.e. is not reduced.

LEMMA 3.1. *Let $G = A * B$ be a free product. The presentation G is aspherical if and only if at least one of the presentations A and B is aspherical.*

PROOF. Remark that the picture over G is decomposed into two part, which are not connected with each other. One corresponds to the picture over the presentation A, and other corresponds to the picture over the presentation B. Thus the picture over G contains a dipole if and only if at least one of pictures, over A or over B, contains a dipole. The lemma is proven.

Thus, below we discuss asphericity only for cases when a group $G_n(m, k)$ is not a free product, or doesn't reduce to groups $G_n(t, 1)$ asphericity of which was investigated in [29]. We will say that a group $G_n(m, k)$ is *strongly irreducible* if defining parameters of it satisfy the following conditions:

$$\begin{cases} 0 < m < k < n, \\ \gcd(n, m, k) = 1, \\ \gcd(n, k) > 1, \\ \gcd(n, k - m) > 1. \end{cases} \tag{3}$$

The following theorem gives a condition of asphericity for strongly irreducible groups.

THEOREM 3.1. *Let $G_n(m, k)$ be a strongly irreducible group. It is aspherical if all of following conditions are not satisfied:*

1) there exists integer $\ell \geq 1$ such that n divides $\ell(2k - m)$ and $\dfrac{1}{\ell} + \dfrac{\gcd(n, k)}{n} + \dfrac{\gcd(n, k - m)}{n} > 1$;

2) $n = k + m$;

3) $n = 2(k - m)$ and $\gcd(n, k) \leq n/2$;

4) $n = 2k$ and $\gcd(n, k - m) < n/2$.

PROOF. Consider the split extension $Y_n(m, k)$ of the group $G_n(m, k)$ by the cyclic group of order n with the generator s acting by the cyclic permutation on the generators $x_1, x_2, \ldots, x_n$ of $G_n(m, k)$, i.e. $s^{-1}x_i s = x_{i+1}$ for $i = 1, \ldots, n$. Therefore, $s^{-i}x_1 s^i = x_{i+1}$. In this case, rewriting the first defining relation of $G_n(m, k)$ in the form $x_{1+m}^{-1}x_1^{-1}x_{1+k} = 1$ we will get $s^{-m}xs^m xs^{-k}x^{-1}s^k = 1$, where $x = x_1^{-1}$. Thus, the group $Y_n(m, k)$ (recall that $0 < m < k < n$) has the presentation

$$Y_n(m, k) = \langle x, s \mid s^n = 1,\ xs^m xs^{-k}x^{-1}s^{k-m} = 1\rangle.$$

Remark that these presentation can be considered as a relative presentation with $H = \langle s|s^n = 1\rangle$.

The following property analogous to Lemma 3.1 from [29] holds.

LEMMA 3.2. *If the presentation $Y_n(m, k)$ is aspherical, then the presentation $G_n(m, k)$ is aspherical too.*

PROOF. Let $\mathcal{P}$ be a picture over the presentation $G_n(m, k)$. Then $\mathcal{P}$ contains discs Δ_i corresponding to relations $x_{i+m}^{-1}x_i^{-1}x_{i+k} = 1$ (see figure 1).

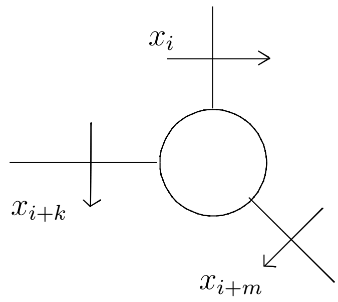

Figure 1.

Let us replace each disc Δ_i by a picture $\mathcal{Q}_i$ over $Y_n(m,k)$ considered as an ordinary (not relative) presentation. The picture $\mathcal{Q}_i$ is presented in figure 2.

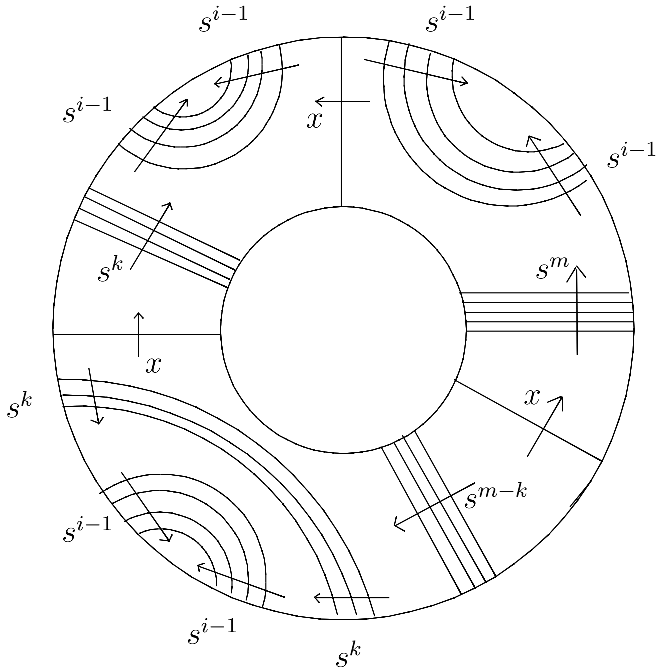

Figure 2.

Here we replaced arcs labeled by x_i, x_{i+m} and x_{i+k} by families of arcs using relations

$$\begin{aligned} x_i &= s^{-(i-1)}x^{-1}s^{i-1}, \\ x_{i+m} &= s^{-(i+m-1)}x^{-1}s^{i+m-1}, \\ x_{i+k} &= s^{-(i+k-1)}x^{-1}s^{i+k-1}, \end{aligned}$$

where $x = x_1^{-1}$. Along the boundary $\partial\mathcal{Q}_i$ we get the relation

$$(s^{-(i+m-1)}xs^{i+m-1})(s^{-(i-1)}xs^{i-1})(s^{-(i+k-1)}x^{-1}s^{i+k-1}) = 1$$

which is equivalent to $x_{i+m}^{-1}x_i^{-1}x_{i+k} = 1$. Along the boundary of the interior disc we get the relation

$$xs^m xs^{-k}x^{-1}s^{k-m} = 1,$$

that is from the presentation $Y_n(m,k)$. Arcs of the picture $\mathcal{Q}_i$ having both ends on $\partial\mathcal{Q}_i$ will give us free circles which can be removed from the picture. We will replace all other arcs with labels s by labels in corners on the disc (see figure 3).

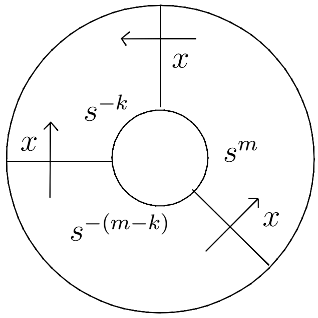

Figure 3.

After repeating the same trick for each disc from the initial picture $\mathcal{P}$ over the presentation $G_n(m,k)$ we will get a picture $\mathcal{Q}$ over the relative presentation $Y_n(m,k)$. By the assumption of asphericity of $Y_n(m,k)$, the picture $\mathcal{Q}$ contains a dipole, i.e. a pair of opposite oriented discs, connected by an arc, which define pairwise inverse words (see figure 4).

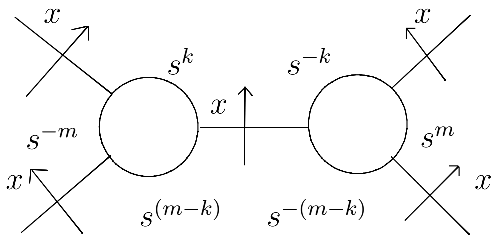

Figure 4.

It is easy to see that by the construction of $\mathcal{Q}$ each such a dipole in $\mathcal{Q}$ arises from a pair of the same, but opposite oriented discs from $\mathcal{P}$ connected by an arc with label x_i for some i. If $\mathcal{Q}$ has a pair of cancelling discs then $\mathcal{P}$ has a pair of cancelling discs too. Therefore the initial picture $\mathcal{P}$ contains a dipole. Thus, any non-empty picture over the presentation $G_n(m,k)$ is equivalent to a picture having two discs less, and so, the presentation is aspherical. Thus, Lemma is proven.

To study asphericity of $Y_n(m,k)$ we will use the following criteria due to Edjvet.

THEOREM 3.2. **[31]** *Let $G = \langle H, x \mid xaxbx^{-1}c\rangle$ for some group H. Suppose that the orders of elements b and c in H are equal to p and q respectively, where $1 < q \le p < \infty$ and $(p,q) \ne (8,4), (9,3)$. Then the group G is aspherical if and only if none of the following hold in H:*
a) $(a^{-1}bac^{-1})^\ell = 1$ for some integer ℓ such that $1/\ell + 1/p + 1/q > 1$,
b) $a^{-1}bac = 1$,
c) $a^{-1}b^2ac = 1$ or $a^{-1}bac^2 = 1$,
d) $q = 2$ and $a^{-1}b^{-1}aca^{-1}bac = 1$,
e) $q = 2$, $p = 3$ and $(a^{-1}bac)^2(a^{-1}b^{-1}ac)^2 = 1$,
f) $p = q = 3$ and $a^{-1}baca^{-1}b^{-1}ac^{-1} = 1$,
g) $q = 3$, $p = 6$ and $a^{-1}b^2ac^{-1} = 1$,
h) $q = p = 7$ and either $a^{-1}b^2ac^{-1} = 1$ or $a^{-1}b^{-1}ac^2 = 1$,
i) $q = p = 9$ and either $a^{-1}b^2ac^{-1} = 1$ or $a^{-1}b^{-1}ac^2 = 1$.

Let us continue the proof of Theorem 3.1.

Remark that the defining relation $xs^mxs^{-k}x^{-1}s^{k-m} = 1$ from the presentation $Y_n(m,k)$ has the form as in the Edjvet's theorem, where $H = \langle s \mid s^n = 1\rangle$ and $a = s^m$, $b = s^{-k}$, $c = s^{k-m}$. In this case order of $b = s^{-k}$ is equal to $n/\gcd(n,k)$ and order of $c = s^{k-m}$ is equal to $n/\gcd(n,k-m)$. Denote

$p = n/\gcd(n,k)$, $q = n/\gcd(n,k-m)$, and suppose that $q \le p$. By the assumption of Theorem 3.1, $p > 1$ and $q > 1$.

In the case a) we have $a^{-1}bac^{-1} = s^{m-2k}$. Since order of s^{2k-m} is equal to $n/\gcd(n,2k-m)$, if for some positive integer ℓ we have $s^{\ell(2k-m)} = 1$, then $n \mid \ell(2k-m)$. Therefore, the condition 1) of Theorem 3.1 holds.

In the case b) we have $a^{-1}bac = s^{-m}$, but because of $0 < m < k < n$, the equality $s^{-m} = 1$ can not hold.

In the case c) there are two possibility. To satisfy the condition $a^{-1}b^2ac = s^{-k-m} = 1$ it must be $n \mid (k+m)$. But since $0 < m < k < n$, it is possible only if $n = k+m$. To satisfy the condition $a^{-1}bac^2 = s^{k-2m} = 1$ it must be $n \mid (k-2m)$. But since $0 < k < m < n$, it is possible only if $k = 2m$, i.e. the group is $G_n(m,2m)$. If $\gcd(n,m) \neq 1$, then the condition $\gcd(n,m,k) = \gcd(n,m,2m) = 1$ is not satisfied. If $\gcd(n,m) = 1$ then the condition $\gcd(n,k-m) = \gcd(n,2m-m) = \gcd(n,m) > 1$ is not satisfied. In both cases the group does not satisfy the assumptions of Theorem 3.1. Thus, the case c) implies the condition 2) of Theorem 3.1.

In the case d) to satisfy the condition $a^{-1}b^{-1}aca^{-1}bac = s^{2(k-m)} = 1$ it must be $n \mid 2(k-m)$. But since $0 < m < k < n$, it is possible only if $n = 2(k-m)$. Obviously, the condition $q = 2$ will be satisfied too. In this case the condition $q \le p$ will become $\gcd(n,k) \le n/2$. Thus, we have got the condition 3) of Theorem 3.1.

Let us consider the case e). Since $p = 3$ and $q = 2$, we get $n = 3\gcd(n,k) = 2\gcd(n,k-m)$. Hence, for some integer $r \ge 1$ we have $n = 6r$ with $\gcd(n,k) = 2r$ and $\gcd(n,k-m) = 3r$. In virtue of the condition $\gcd(n,m,k) = 1$ it is possible only if $r = 1$. Using that $0 < m < k < n$, we conclude $n = 6$, $k = 4$ and $m = 1$, that is a particular case of the condition 3) of Theorem 3.1.

The last cases f)–i) of Theorem 3.2 can not arise in our case. Indeed, if $p = uq$ for some integer $u \ge 1$, then $u\gcd(n,k) = \gcd(n,k-m)$. Denoting $v = \gcd(n,k)$, we get $v \mid n$ and $v \mid k$, where in virtue of assumptions of Theorem 3.1 we have $v > 1$. Since $v \mid \gcd(n,k-m)$, then $v \mid m$, that gives the contradiction with $\gcd(n,m,k) = 1$.

Remark that above considered cases corresponded to the assumption $q \le p$, i.e. to the case $n/\gcd(n,k) \le n/\gcd(n,k-m)$.

Now let us consider the situation when $n/\gcd(n,k) > n/\gcd(n,k-m)$. Remark that the relation $xs^mxs^{-k}x^{-1}s^{k-m} = 1$ from $Y_n(m,k)$ is equivalent to the relation $zs^{-m}zs^{m-k}z^{-1}s^k = 1$, where $z = x^{-1}$. This word is as in the Edjvet's theorem, where $a = s^{-m}$, $b = s^{m-k}$ and $c = s^k$. In this case order of b is equal to $n/\gcd(n,k-m)$ and order of c is equal to $n/\gcd(n,k)$. Denoting $p = n/\gcd(n,k)$ and $q = n/\gcd(n,k-m)$ we get $q < p$, and assumptions of the Edjvet's theorem are satisfied.

Similar to the above considered situation, cases a), b) and c) give us conditions 1) and 2) of Theorem 3.1.

In the case d) to satisfy the condition $a^{-1}b^{-1}aca^{-1}bac = s^{2k} = 1$ it must be $n \mid 2k$. Since $1 < k < n$, it is possible only if $n = 2k$. Obviously, the condition $q = 2$ will be satisfied too. The condition $q \le p$ will give us $\gcd(n,k-m) < n/2$. Hence, we have got the condition 4) of Theorem 3.1.

Consider the case e). Since $p = 3$ and $q = 2$, we get $n = 3\gcd(n,k-m) = 2\gcd(n,k)$. Therefore, for some integer $r \ge 1$ we have $n = 6r$, $\gcd(n,k) = 3r$ and $\gcd(n,k-m) = 2r$. In virtue of the condition $\gcd(n,m,k) = 1$ it is possible only if $r = 1$. Since $0 < m < k < n$, we get $n = 6$, $k = 3$ and $m = 1$, that is the particular case of the condition 4) of Theorem 3.1.

The last cases f)–i) of Theorem 3.2 can not arise in our case by the same arguments as in the previous situation.

Summarizing the above considered cases and applying Lemma 3.2 we get the statement of Theorem 3.1.

4. Groups with odd number of generators

In [1] the following question was posed: for which values of n, m and k groups $G_n(m,k)$ are fundamental groups of 3-manifolds? As it was observed in Section 2, groups $G_n(m,k)$ contains Fibonacci groups and Sieradski groups for which corresponding 3-manifolds were described in [21] and [27].

The following result, which is a generalization of the result from [24] for Fibonacci groups, shows that in many cases the groups under consideration can not be fundamental groups of hyperbolic 3-manifolds (see for example [37] about geometrical structures on 3-manifolds).

THEOREM 4.1. *Let n be odd, $k-m$ even, and $\gcd(m-2k,n)=1$. Then the group $G_n(m,k)$ can not be group of a hyperbolic 3-orbifold (in particular, 3-manifold) of finite volume.*

PROOF. We use the same method for the proof as in [24, 36]. Let us assume, controversy, that $G_n(m,k)$ is group of a hyperbolic 3-orbifold of finite volume. Therefore, it can be realized as a group of isometries of the three-dimensional Lobachevsky space $\mathbf{H}^3$, and moreover, it is a crystallographic group of motions [39]. Consider the split extension $W_n(m,k)$ of $G_n(m,k)$ by the cyclic automorphism s such that $s: x_i \to x_{i+1}$ for $i=1,2,\ldots,n$. Denote its order by n_1. Obviously, $n_1 \mid n$. Recall [39, p. 231] that any isomorphism of crystallographic groups of motions of the Lobachevsky space is induced by a conjugation in the group of isometries of this space. Let us denote this conjugating isometry by s also. Suppose

$$W_n(m,k) = \langle G_n(m,k),\ s\rangle = \ \langle x_1,\ldots,x_n, s \mid$$

$$s^{n_1}=1,\ x_1 s^{-m} x_1 s^m = s^{-k} x_1 s^k,\ x_{i+1} = s^{-1} x_i s,\ i=1,\ldots,n\rangle.$$

Using Tits transformations, we will get

$$W_n(m,k) = \langle x, s \mid s^{n_1}=1,\, x s^{-m} x s^m = s^{-k} x s^k\rangle,$$

where $x=x_1$. Let us consider the verbal subgroup of $W=W_n(m,k)$ generated by squares of elements, $W^2=\langle w^2 \mid w\in W\rangle$. From the second defining relation of the group W we get

$$s^k x s^{-m} = x s^{k-m} x^{-1}.$$

Since $k-m$ is even, $s^{k-m}\in W^2$. Therefore, $xs^{k-m}x^{-1}\in W^2$, and then $s^k x s^{-m}\in W^2$, i.e. $x\in W^2$. Since n_1 is odd, we have $s\in W^2$. Thus, the group W (which is supposed to be the group of a hyperbolic 3-orbifold) contains only orientation preserving isometries of the Lobachevsky space. Recall [37], that the full group of orientation preserving isometries of the Lobachevsky is isomorphic to $PSL_2(\mathbf{C})$. Whence, W is a subgroup of $PSL_2(\mathbf{C})$.

Since W is a subgroup of $PSL_2(\mathbf{C})$, there exists a subgroup $\widetilde{W}$ of $SL_2(\mathbf{C})$ which is a preimage of W in respect to the canonical projection. Suppose that for the generator x of W the corresponding element in $\widetilde{W}$ is presented by a matrix $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL_2(\mathbf{C})$. Since W uniformizes a hyperbolic 3-orbifold of finite volume, it is not elementary (in sense of Kleinian group theory [38]) and so, $\beta\neq 0$ and $\gamma\neq 0$.

For the generator $s\in W$ the corresponding matrix in $\widetilde{W}$ is $\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}\in SL_2(\mathbf{C})$, where ζ is a primitive root from 1 of degree $2n_1$. Then the relation

$$x s^{-m} x s^m = s^{-k} x s^k$$

induces the following relation for matrices:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \zeta^{-m} & 0 \\ 0 & \zeta^{m} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \zeta^{m} & 0 \\ 0 & \zeta^{-m} \end{pmatrix} = \begin{pmatrix} \zeta^{-k} & 0 \\ 0 & \zeta^{k} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \zeta^{k} & 0 \\ 0 & \zeta^{-k} \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix},$$

where $\varepsilon = \pm 1$. Multiplying matrices in left and right parts we get

$$\begin{pmatrix} \alpha^2 + \beta\gamma\zeta^{2m} & \beta(\alpha\zeta^{-2m} + \delta) \\ \gamma(\alpha + \delta\zeta^{2m}) & \beta\gamma\zeta^{-2m} + \delta^2 \end{pmatrix} = \begin{pmatrix} \varepsilon\alpha & \varepsilon\beta\zeta^{-2k} \\ \varepsilon\gamma\zeta^{2k} & \varepsilon\delta \end{pmatrix}.$$

Hence, using $\beta \neq 0$ and $\gamma \neq 0$, we have

$$\begin{cases} \alpha + \delta\zeta^{2m} = \varepsilon\zeta^{2k} \\ \alpha\zeta^{-2m} + \delta = \varepsilon\zeta^{-2k} \end{cases}.$$

Multiplying the second equation by ζ^{2m} we will get the equivalent system:

$$\begin{cases} \alpha + \delta\zeta^{2m} = \varepsilon\zeta^{2k} \\ \alpha + \delta\zeta^{2m} = \varepsilon\zeta^{2(m-k)} \end{cases}.$$

From this system we get $\zeta^{2(m-2k)} = 1$. At the same time, $\zeta^{2n_1} = 1$. Since we assumed that $\gcd(n, m-2k) = 1$, we have got a contradiction. Therefore, $G_n(m,k)$ can not be the group of a hyperbolic 3-orbifold of finite volume. Theorem is proven.

Recall that groups $G_n(1,2)$ with even number of generators $n \geq 8$ are fundamental groups of hyperbolic 3-manifolds and, therefore, don't contain elements of finite order. The following statement demonstrate that for the case of odd n the situation is different.

PROPOSITION 4.1. *Group $G_n(m,2m)$ with odd number of generators has a torsion.*

PROOF. By Corollary 2.2 $G_n(m,2m)$ is either isomorphic to the Fibonacci group $G_n(1,2)$ or decomposes in a free product of Fibonacci group with less number of generators (and for each of them the number of generators is also odd). Thus, it is enough to discuss only groups $G_n(1,2)$.

Consider in $G_n(1,2)$ the element $w = \prod_{i=0}^{2n-1} x_{1+i}$, where all indices are taken by mod n. We claim that this element is of order two. Indeed, presenting w in the form $w = uv$, where

$$u = \prod_{i=0}^{n-1} x_{1+im} = \prod_{i=0}^{2n_1} x_{1+im} \quad \text{and} \quad v = \prod_{i=n}^{2n-1} x_{1+i} = \prod_{i=0}^{n-1} x_{1+(n+i)} = \prod_{i=0}^{2n_1} x_{1+n+i} = \prod_{i=0}^{2n_1} x_{1+i} = u,$$

since all indices are by mod n, we get $w = u^2$.

At the same time, in virtue of the relation $x_i x_{i+1} = x_{i+2}$, which can be rewritten in the form $x_{i+1} = x_i^{-1} x_{i+2}$, we have

$$w = \prod_{i=0}^{2n-1} x_{1+i} = \prod_{i=0}^{4n_1+1} x_{1+i} = \prod_{i=0}^{2n_1} (x_{1+2i} x_{2+2i}) = \prod_{i=0}^{2n_1} x_{3+2i} = \prod_{i=0}^{2n_1} \left(x_{2+2i)}^{-1} x_{4+2i} \right) =$$

$$= x_2^{-1} x_{4+4n_1} = x_2^{-1} x_{2+2(2n_1+1)} = x_2^{-1} x_{2+2n} = x_2^{-1} x_2 = 1.$$

Therefore, $u^2 = 1$. Essentially, the idea to consider the element of such form is due to [6].

Now we will show that u is trivial if and only if for each $j = 1, \dots, n$ generators x_j and x_{j-1} commutate. Indeed, the condition $u = 1$ means that for each $j = 1, \dots, n$ the following relation holds:

$$x_j \, x_{j+1} \, x_{j+2} \, \cdots \, x_{j+(2n_1-2)} \, x_{j+(2n_1-1)} \, x_{j+2n_1} = 1.$$

Using relation $x_i x_{i+1} = x_{i+2}$ for different indices, we will get

$$x_{j+2}^2 \, x_{j+3} \, \cdots \, x_{j+(2n_1-2)} \, x_{j+(2n_1-1)} \, x_{j+2n_1} = 1,$$

whence

$$x_{j+2} \, x_{j+4}^2 \, \cdots \, x_{j+(2n_1-2)} \, x_{j+(2n_1-1)} \, x_{j+2n_1} = 1,$$

and then,

$$x_{j+2} \, x_{j+4} \, \cdots \, x_{j+(2n_1-2)} \, x_{j+2n_1}^2 = 1.$$

Whence,

$$\left(x_{j+1}^{-1} x_{j+3}\right) \, \left(x_{j+3}^{-1} x_{j+5}\right) \, \cdots \, \left(x_{j+(2n_1-1)}^{-1} x_{j+(2n_1+1)}\right) \, x_{j+2n_1} = 1$$

and therefore,

$$x_{j+1}^{-1} \, x_{j+2n_1+1} \, x_{j+2n_1} = 1.$$

Since all indices are taken by mod $n = 2n_1 + 1$, it gives $x_j \, x_{j-1} = x_{j+1}$. Recalling that $x_{j-1} \, x_j = x_{j+1}$ we get that for each $j = 1, \dots, n$ generators x_j and x_{j-1} commutate.

Remark, that it implies that any two generators, say x_i and x_{i+p}, also commutate. For $p = 1$ it was just shown. Suppose that for all q such that $q < p$ generators x_i and x_{i+q} commutate. Then

$$x_i x_{i+p} = x_i x_{i+p-2} x_{i+p-1} = x_{i+p-2} x_i x_{i+p-1} = x_{i+p-2} x_{i+p-1} x_i = x_{i+p} x_i.$$

Thus, u is trivial if and only if the group $G_n(1,2)$ is abelian. Recall that if n is odd, the Fibonacci group $F(2,n) = G_n(1,2)$ is finite for $n = 1, 3, 5, 7$ (and so, has a torsion), and is infinite for $n \geq 9$ (see [35]). In these cases, since the abelizator is finite (see [6]), the group can be abelian. Thus, u is non-trivial element of order two. The Proposition is proven.

Since in the above proof the commutativety was the crucial point, we recall that due to Reidemeister (see, for example, [3, ch. 5]) the full list of abelian group that can arise as fundamental groups as 3-manifolds is the following: $\mathbf{Z}_n$, $\mathbf{Z}$, $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$, $\mathbf{Z} \oplus \mathbf{Z}_2$.

5. On a finiteness of groups $G_n(m,k)$

Recall that the investigation of the Fibonacci groups $F(2,n) \cong G_n(1,2)$ was initiated in 1965 by the question of Conway [5] about finiteness of the group $F(2,5)$. The survey of results about the Fibonacci groups $F(2,n)$ is done in [9]: these groups are finite if and only if $n = 1, 2, 3, 4, 5, 7$. Moreover, $F(2,1) \cong F(2,2) \cong \langle \mathbf{1} \rangle$, $F(2,3) \cong Q_8$ that is the quaternion group of order 8, $F(2,4) \cong \mathbf{Z}_5$, $F(2,5) \cong \mathbf{Z}_{11}$, $F(2,7) \cong \mathbf{Z}_{29}$. The proof of infinitely of groups $F(2,n)$ for even $n \geq 6$ can be found in [21] or [8], and for odd $n \geq 9$ in [35].

Considering the Cavicchioli-Hegenbarth-Repovš groups, the following question naturally arises:

QUESTION 1. *For which values of defining parameters n, m, k with a natural assumptions $0 < m < k < n$ and $\gcd(n,m,k) = 1$ groups $G_n(m,k)$ are finite?*

The computer program GAP [32] is useful to find orders of groups $G_n(m,k)$ for small values of n.

For $n = 3$ there is only one group $G_3(1,2) \cong F(2,3)$ that is the quaternion group of order 8. For $n = 4$ we have $G_4(1,3) \cong G_4(1,2) \cong F(2,4) \cong \mathbf{Z}_5$ (see Example 2.1), $|G_4(2,3)| = 24$, $|G_4^{\mathrm{ab}}(2,3)| = 3$, $G_4(2,3)$ is a 3-step-solvable. Other results of computations are presented in the tables below.

n=5	m \ k	2	3	4
	1	11	120	11
	2	–	11	11
	3	–	–	120

n=6	m \ k	2	3	4	5
	1	∞	7	7	∞
	2	–	9	∞	9
	3	–	–	56	56
	4	–	–	–	∞

n=7	m \ k	2	3	4	5	6
	1	29	?	?	?	29
	2	–	?	29	29	?
	3	–	–	29	?	29
	4	–	–	–	?	?
	5	–	–	–	–	?

n=8	m \ k	2	3	4	5	6	7
	1	∞	295245	17	17	295245	?
	2	–	?	∞	?	∞	?
	3	–	–	17	?	∞	17
	4	–	–	–	8	∞	?
	5	–	–	–	–	295245	295245
	6	–	–	–	–	–	?

In these tables the intersection of m's row and k's column gives order of the group $G_n(m,k)$. The sign "–" means that parameters m and k do not satisfy the condition $m < k$.

The group $G_6(1,2)$ is the Fibonacci group $F(2,6)$ and so is infinite [21], $G_6(2,4) \cong G_3(1,2) * G_3(1,2)$ is also infinite. Next we have that $G_8(1,2) \cong F(2,8) \cong G_8(3,6)$ is infinite, $G_8(2,4) \cong G_4(1,2)*G_4(1,2)$, $G_8(2,6) \cong G_4(1,3)*G_4(1,3)$, $G_8(4,6) \cong G_4(2,3)*G_4(2,3)$. The group $G = G_6(3,4)$ is metabelian, i.e. its second derived group is trivial $G'' = 1$ and $|G'| = 8$, $G/G' \cong \mathbf{Z}_7$. The group $G_5(1,3)$ coincides with its derived group. The group $G_6(4,5) \cong G_6(2,1)$ has infinite abelizator [30], so it is infinite.

Next one can ask about isomorphisms of groups.

QUESTION 2. *Describe the function $f(n)$ which for an integer $n \geq 3$ gives the number of pairwise non-isomorphic groups $G_n(m,k)$, where $0 < m < k < n$.*

From the above considerations we have $f(3) = 1$, $f(4) = 2$ and $f(5) = 2$.

For $n = 6$ from Theorem 2.1 and the table of abelizators below, we see that there are 6 classes of pairwise isomorphic groups: $G_6(1,2) \cong G_6(1,5)$, $G_6(1,3) \cong G_6(1,4)$, $G_6(2,3) \cong G_6(2,5)$, $G_6(2,4)$, $G_6(3,4) \cong G_6(3,5)$, $G_6(4,5)$. Hence $f(6) = 6$.

For $n = 7$ from Theorem 2.1 and the table of abelizators below, we see that there are 3 classes of pairwise isomorphic groups:

1) $\{G_7(1,2), G_7(1,6), G_7(2,4), G_7(2,5), G_7(3,4), G_7(3,6)\}$,
2) $\{G_7(1,3), G_7(1,5), G_7(2,3), G_7(2,6), G_7(4,5), G_7(4,6)\}$,
3) $\{G_7(1,4), G_7(3,5), G_7(5,6)\}$.

Therefore, $f(7) = 3$.

n=6	m \ k	2	3	4	5
	1	$\mathbf{Z}_4^2$	$\mathbf{Z}_7$	$\mathbf{Z}_7$	$\mathbf{Z}_4^2$
	2	–	$\mathbf{Z}_9$	$\mathbf{Z}_2^4$	$\mathbf{Z}_9$
	3	–	–	$\mathbf{Z}_7$	$\mathbf{Z}_7$
	4	–	–	–	$\mathbf{Z}^2$

n=7	m \ k	2	3	4	5	6
	1	$\mathbf{Z}_{29}$	$\mathbf{Z}_2^3$	**1**	$\mathbf{Z}_2^3$	$\mathbf{Z}_{29}$
	2	–	$\mathbf{Z}_2^3$	$\mathbf{Z}_{29}$	$\mathbf{Z}_{29}$	$\mathbf{Z}_2^3$
	3	–	–	$\mathbf{Z}_{29}$	**1**	$\mathbf{Z}_{29}$
	4	–	–	–	$\mathbf{Z}_2^3$	$\mathbf{Z}_2^3$
	5	–	–	–	–	**1**

6. Some open questions

In conclusion we formulate some open questions, additionally to pointed out above, about Cavicchioli-Hegenbarth-Repovš groups.

QUESTION 3. *Find rank of a group* $G_n(m,k)$, $0 \leq m < k < n$.

QUESTION 4. *For which values of* n, m, k *groups* $G_n(m,k)$ *are linear?*

QUESTION 5. *Can groups* $G_n(m,k)$ *and* $G_{n'}(m',k')$ *be isomorphic for* $n \neq n'$*?*

It is well-known [6] that abelizator of the Fibonacci group $F(2,n)$ is finite and its order is equal to $f_n - 1 - (-1)^n$, where f_n is a Fibonacci number.

QUESTION 6. *If there is a similar formula which gives the order of the abelizator of* $G_n(m,k)$ *when t is finite? Can such kind of a formula be given in terms of numbers generalizing the Fibonacci numbers?*

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