Linear representations of the braid groups of some manifolds *

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Abstract

In this paper we prove that the braid group $B_n(S^2)$ of 2–sphere, mapping class group $M(0,n)$ of the n–punctured 2–sphere and the braid group $B_3(P^2)$ of the projective plane are linear.

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1 Introduction

One of the most intriguing problems in the theory of braid groups is the question of whether B_n is linear, i. e., whether it admits a faithful representation into a group of matrices over the field. This question was formulated by W. Burau [1] in 1936 who found that n–dimensional linear representation of B_n which for a long time had been considered as a candidate for faithful representation. However, as it was established by J. Moody [2] in 1991 this representation is not faithful for $n \geq 9$. This bound was improved for $n \geq 6$ by D. Long and M. Paton [3] and to $n \geq 5$ by S. Bigelow [4]. Until now, it is unknown whether the Burau representation of B_4 is faithful.

In 1990 R. Lawrence [5] introduced a family of linear representations of B_n. Later D. Krammer [6] and S. Bigelow [7] proved that one of this representation is faithful. Therefore B_n are linear for all $n \geq 2$.

Braid group B_n is a particular case of general construction of braid group $B_n(M)$ of manifold M on n strings.

Question 1 *For which manifold M and natural n is the braid group $B_n(M)$ on n strings linear?*

The greatest interest in this problem is connected to manifolds of dimension two. In the braid groups of these manifolds only $B_n(S^2)$ of 2–sphere S^2 and $B_n(P^2)$ of projective plane P^2 have a torsion.

In this article we will prove that the mapping class group $M(0,n)$ of the n–punctured 2–sphere and braid group $B_n(S^2)$ of 2–sphere are linear for all $n \geq 2$. Also we will prove that braid group $B_3(P^2)$ of projective plane is linear (for $n = 1, 2$ this group is finite). In all cases we will construct

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corresponding linear representation. The result that $B_n(S^2)$ is linear was announced in [8]. The author awares that S. Bigelow and R. D. Budney [9] give another proof that groups $B_n(S^2)$ and $M(0,n)$ are linear.

2 Definitions and notations

We remind you of some known facts which can be found in [10, 11, 12].

The group G is a *semi direct product* of A and B if there exist subgroups H and K in G such that

$$G = HK, \quad A \simeq H \trianglelefteq G, \quad B \simeq K, \quad H \cap K = 1.$$

Semi direct product is denoted as $G = A \leftthreetimes B$. If A and B have presentations

$$A =< a_1, a_2, \dots, a_k \parallel A_1, A_2, \dots, A_p >, \quad B =< b_1, b_2, \dots, b_l \parallel B_1, B_2, \dots, B_q >,$$

then $G = A \leftthreetimes B$ have presentation

$$G =< a_1, a_2, \dots, a_k; b_1, b_2, \dots, b_l \parallel A_1, A_2, \dots, A_p; B_1, B_2, \dots, B_q; b_i^{-1} a_j b_i = C_{ij},$$

$$1 \leq i \leq l, \quad 1 \leq j \leq k >,$$

where $A_1, A_2, \dots, A_p, C_{ij}$ are words in alphabet $\{a_1^{\pm 1}, a_2^{\pm 1}, \dots, a_k^{\pm 1}\}$, $B_1, B_2, \dots, B_q$ are words in alphabet $\{b_1^{\pm 1}, b_2^{\pm 1}, \dots, b_l^{\pm 1}\}$ and elements $C_{i1}, C_{i2}, \dots, C_{ik}$ generate group A for each $i \in \{1, 2, \dots, l\}$.

Let M be a manifold of dimension ≥ 2. *Configuration space* $F_n(M)$, $n \in \mathbb{N}$, for manifold M is a set

$$F_n(M) = \{(z_1, z_2, \dots, z_n) \in M^n | z_i \neq z_j \ \text{ for } \ i \neq j\}$$

of ordering collections of n distinct points from M. The fundamental group $P_n(M) = \pi_1(F_n(M))$ of the space $F_n(M)$ is the *pure braid group* with n strings of the manifold M. Symmetric group S_n acts on the space $F_n(M)$ by permuting the coordinates. This action is free and induce the regular covering of orbit space $F_n(M)/S_n$ by $F_n(M)$. The fundamental group $B_n(M) = \pi_1(F_n(M)/S_n)$ is called the *full braid group* of M, or just the *braid group* of M. The regular covering projection $F_n(M) \longrightarrow F_n(M)/S_n$ induces the short exact sequence

$$1 \longrightarrow P_n(M) \longrightarrow B_n(M) \longrightarrow S_n \longrightarrow 1.$$

If M is a closed, smooth manifold of dimension $n \geq 2$, then the inclusion map $F_n(M) \longrightarrow M^n$ induces a surjective homomorphism $P_n(M) \longrightarrow \pi_1(M) \times \ldots \times \pi_1(M)$ pure braid group $P_n(M)$ on direct product n copies of fundamental group $\pi_1(M)$. If dim$M > 2$ then this homomorphism is also injective. The braid groups of manifolds of dimension 2 represent the largest interest.

The 2–sphere S^2 and projective plane P^2 play a special role among the 2–dimension manifolds because the braid groups only of this manifolds have elements of finite order. If M is a closed surface different from S^2 or P^2, then in the following sequence of groups

$$1 \longrightarrow P_n(E^2) \longrightarrow P_n(M) \longrightarrow \prod_{i=1}^{n} \pi_1(M) \longrightarrow 1$$

the kernel of each homomorphism is equal to the normal closure of the image of the previous homomorphism in the sequence. In this sequence E^2 denotes the Euclidean plane and $\prod_{i=1}^{n} \pi_1(M)$ denotes the direct product of n copies of $\pi_1(M)$.

The classical braid group of Artin B_n is the braid group $B_n(E^2)$ of Euclidean plane E^2. We will call B_n just a braid group.

The braid group B_n, $n \geq 2$, with n strings can be defined as the group generated by $n-1$ generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ with defining relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, 2, ..., n-2, \tag{1}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| \geq 2. \tag{2}$$

There is a homomorphism $\nu : B_n \longrightarrow S_n$ from the group B_n to the symmetric group S_n on n letters defined by

$$\nu(\sigma_i) = (i, i+1), \quad i = 1, 2, \ldots, n-1.$$

The kernel of homomorphism ν is the pure braid group P_n. The group P_n admits a presentation with generators

$$a_{i,i+1} = \sigma_i^2,$$

$$a_{ij} = \sigma_{j-1} \sigma_{j-2} \ldots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \ldots \sigma_{j-2}^{-1} \sigma_{j-1}^{-1}, \quad i+1 < j \leq n.$$

and defining relations:

$$a_{ik}^{-\nu} a_{kj} a_{ik}^{\nu} = \left(a_{ij} a_{kj}\right)^{\nu} a_{kj} \left(a_{ij} a_{kj}\right)^{-\nu},$$

$$a_{km}^{-\nu} a_{kj} a_{km}^{\nu} = \left(a_{kj} a_{mj}\right)^{\nu} a_{kj} \left(a_{kj} a_{mj}\right)^{-\nu}, \qquad m < j,$$

$$a_{im}^{-\nu} a_{kj} a_{im}^{\nu} = \left[a_{ij}^{-\nu}, a_{mj}^{-\nu}\right]^{\nu} a_{kj} \left[a_{ij}^{-\nu}, a_{mj}^{-\nu}\right]^{-\nu}, \qquad i < k < m,$$

$$a_{im}^{-\nu} a_{kj} a_{im}^{\nu} = a_{kj}, \qquad k < i; \quad m < j \ \text{ or } \ m < k,$$

where $\nu = \pm 1$.

The subgroup U_n of P_n which is generated by $a_{1,n}, a_{2,n}, \ldots, a_{n-1,n}$ is free and normal in P_n. The group P_n is a semi direct product of U_n and P_{n-1}. Hence the group P_n is a semi direct product

$$P_n = U_n \leftthreetimes (U_{n-1} \leftthreetimes (\ldots \leftthreetimes (U_3 \leftthreetimes U_2)) \ldots),$$

where $U_i = < a_{1,i}, a_{2,i}, \ldots, a_{i-1,i} >, i = 2, 3, \ldots, n$, is a free group of rank $i-1$.

We remind you of the definition (see [5, 6, 7]) of faithful linear representation of Lawrence–Krammer braid group B_n. Let V_n be a free module dimension $m = n(n-1)/2$ with basis v_{ij}, $1 \leq i < j \leq n$, over ring $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ of Laurent polynomials on two variables. Then representation $\rho : B_n \longrightarrow \mathrm{GL}(V_n)$ is defined by action by σ_i, $i = 1, 2, \ldots, n-1$, on module V_n by equalities (we will write $\sigma_i(v_{j,k})$ or $v_{j,k}^{\sigma_i}$ instead of $\rho(\sigma_i)(v_{j,k})$ or $v_{j,k}^{\rho(\sigma_i)}$ accordingly):

$$\sigma_i(v_{k,i}) = (1-q)v_{k,i} + qv_{k,i+1} + q(q-1)v_{i,i+1},$$

$$\sigma_i(v_{k,i+1}) = v_{k,i} \quad \text{if} \quad k < i,$$

$$\sigma_i(v_{i,i+1}) = tq^2 v_{i,i+1},$$

$$\sigma_i(v_{i,l}) = tq(q-1)v_{i,i+1} + (1-q)v_{i,l} + qv_{i+1,l} \quad \text{if} \quad i+1 < l,$$

$$\sigma_i(v_{i+1,l}) = v_{i,l},$$

$$\sigma_i(v_{k,l}) = v_{k,l} \quad \text{if} \quad \{k,l\} \bigcap \{i,i+1\} = \emptyset.$$

3 Linear representations

As it was proved (see [11, theorem 1.11; 13]) the braid group $B_n(S^2)$ of the 2–sphere S^2 admits a presentation with generators $\delta_1, \delta_2, \ldots, \delta_{n-1}$ and defining relations:

$$\delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1} \quad \text{if} \quad i = 1, 2, ..., n-2,$$

$$\delta_i\delta_j = \delta_j\delta_i \quad \text{if} \quad |i-j| \geq 2,$$

$$\delta_1\delta_2 \ldots \delta_{n-2}\delta_{n-1}^2\delta_{n-2} \ldots \delta_2\delta_1 = 1.$$

¿From this relations we see that $B_n(S^2)$ is a homomorphic image of B_n. Since $B_2(S^2)$ is a cyclic group of order 2 and $B_3(S^2)$ is a metacyclic group of order 12, then we will consider $n > 3$.

R. Gillette and J. Van Buskirk [13] have studied the structure of $B_n(S^2)$. We recall some of their results. Let

$$a_{i,i} = 1, \quad a_{i,j} = \delta_i^{-1}\delta_{i+1}^{-1} \ldots \delta_{j-2}^{-1}\delta_{j-1}^2\delta_{j-2} \ldots \delta_{i+1}\delta_i, \quad 1 \leq i < j \leq n. \tag{1}$$

These elements generate the pure braid group $P_n(S^2)$ which is a kernel of homomorphism $\nu : B_n(S^2) \longrightarrow S_n$ and ν send δ_i in transposition $(i, i+1)$, $1 \leq i \leq n-1$, from S_n. Let define the subgroup $A_{n-i+1} =< a_{i,i+1}, a_{i,i+2}, \ldots, a_{i,n} >$ for each $i = 1, 2, \ldots, n-1$. Subgroup A_n is normal in $P_n(S^2)$ and we have the short exact sequence

$$1 \longrightarrow A_n \longrightarrow P_n(S^2) \longrightarrow P_{n-1}(S^2) \longrightarrow 1$$

and $P_n(S^2)$ is a semi direct product: $P_n(S^2) = A_n \leftthreetimes P_{n-1}(S^2)$. The generators of A_n are connected by the relation

$$a_{1,2}a_{1,3} \ldots a_{1,n} = 1.$$

Since the following relations are held in $P_n(S^2)$

$$a_{i,i+1}a_{i,i+2} \ldots a_{i,i+n-1} = 1, \quad i = 1, 2, \ldots, n-1,$$

where $a_{j,i} = a_{i,j}$ if $j > i$ and indices are taken mod$n$, then

$$a_{i,n} = a_{i,n-1}^{-1} \ldots a_{i,i+1}^{-1}a_{i-1,i}^{-1} \ldots a_{1,i}^{-1}, \quad i = 1, 2, \ldots, n-1.$$

Using these formulas we can exclude $a_{1,n}, a_{2,n}, \dots, a_{n-1,n}$ from the set of generators of $P_n(S^2)$. Hence A_{n-i+1} is freely generated by $a_{i,i+1}, a_{i,i+2}, \dots, a_{i,n-1}$.

The center of $B_n(S^2)$ is the cyclic group of order 2 generated by the Dirac braid

$$\Delta_n = (\delta_1 \delta_2 \dots \delta_{n-2})^{n-1} = (a_{1,2} a_{1,3} \dots a_{1,n-1})(a_{2,3} a_{2,4} \dots a_{2,n-1}) \dots (a_{n-2,n-1}).$$

The group $P_n(S^2)$ was splitted in the semi direct product

$$P_n(S^2) = A_n \leftthreetimes (A_{n-1} \leftthreetimes (\dots \leftthreetimes A_3) \dots).$$

Let $L_n = A_n \leftthreetimes (A_{n-1} \leftthreetimes (\dots \leftthreetimes A_4) \dots)$. Since A_3 is a cyclic group generated by $a_{n-2,n-1}$, then $P_n(S^2) = L_n \times < \Delta_n > \simeq L_n \times \mathbb{Z}_2$. Group L_n is isomorphic to subgroup $U_n \leftthreetimes (U_{n-1} \leftthreetimes (\dots \leftthreetimes U_4) \dots) \leq P_n$.

The braid group of sphere closed relates with the group $M(0, n)$ of mapping classes of the n–punctured 2–sphere. Presentation of the $M(0, n)$ is derived from that of $B_n(S^2)$ by adding the relation: $\Delta_n = (\delta_1 \delta_2 \dots \delta_{n-2})^{n-1} = 1$. There is an epimorphism from $B_n(S^2)$ to $M(0, n)$. The kernel of this epimorphism is equal to the center of $B_n(S^2)$. A mapping class of the n–punctured 2–sphere can be visualized as an n–string braid between concentric 2–spheres, where the inner sphere is free to execute full revolutions.

For $M(0, n)$ there is a short exact sequence

$$1 \longrightarrow L_n \longrightarrow M(0, n) \longrightarrow S_n \longrightarrow 1,$$

where L_n is a kernel of epimorphism $\nu : M(0, n) \longrightarrow S_n$ that map δ_i to the transposition $(i, i+1)$, $i = 1, 2, \dots, n-2$.

A. I. Malcev [14, lemma 1; 10] has proved that if H is a subgroup of finite index in group G and H is linear, then G is linear too. Recall his construction. Let $|G : H| = m$ and $\psi : H \longrightarrow \mathrm{GL}_l(F)$ is a faithful linear representation of H by matrices of order l over field F. In this case G is a union of right cosets

$$G = He \bigcup Hg_2 \bigcup \dots \bigcup Hg_m.$$

For each $g \in G$ we can write the product $g_i g$ uniquely in the form $h_i g_{n_i}$, $h_i \in H$. Therefore for each g we have the sequence $h_1, h_2, \dots, h_m$ from H and the sequence numbers $n_1, n_2, \dots, n_m$. For the sequence $\{n_i\}$ we construct the matrix $D(n_i)$ with interger coefficients by the rule:

$$D(n_i) = \| d_{j,k} \| \in M_{lm}(\mathbb{Z}), \quad d_{j,n_j} = E_l, \quad d_{j,k} = 0 \quad \text{if} \quad k \neq n_j,$$

where E_l is identical matrix of order l. Then we can map $g \in G$ to the matrix

$$\mathrm{diag}(\psi(h_1), \psi(h_2), \dots, \psi(h_m)) D(n_i).$$

Hence we constructed the linear representation of G in $\mathrm{GL}_{lm}(F)$. This linear representation is faithful. Using this construction we will prove

Theorem 3.1 *Let $R = \mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ be a ring of Laurent polynomials on two variables. Then the groups $M(0, n)$ and $B_n(S^2)$ are linear for each $n \geq 2$. For each $n \geq 4$ there are inclusion maps $\varphi : M(0, n) \longrightarrow \mathrm{GL}_m(R)$ and $\varphi_1 : B_n(S^2) \longrightarrow \mathrm{GL}_{m_1}(R)$, where $m = (n-1)(n-2)n!/2$, $m_1 = 2m$.*

Proof: As we noted above, if $n = 2, 3$ groups $M(0, n)$, $B_n(S^2)$ are finite and hence are linear. Since L_n is isomorphic to subgroup of B_n, then is a faithful linear representation $\rho : L_n \longrightarrow \mathrm{GL}_l(R)$

for $l = (n-1)(n-2)/2$. This representation is induced by Lawrence–Krammer representation of B_n.

Let $m_1, m_2, \ldots, m_{n!}$ be coset representatives of $M(0,n)$ by subgroup L_n. Since $M(0,n)$ is generated by $\delta_1, \delta_2, \ldots, \delta_{n-1}$ then we can define φ only on these elements. Each generator δ_k acts on set of coset representatives by permutation. We find

$$m_i \delta_k = h_i^k m_{\pi_k(i)}, \quad i = 1, 2, \ldots, n!, \quad k = 1, 2, \ldots, n-1,$$

where symbol k in the upper part is an index but not an exponent (this rule will be true to the end of this proof). We compare δ_k with the matrix

$$\varphi(\delta_k) = \mathrm{diag}(\rho(h_1^k), \rho(h_2^k), \ldots, \rho(h_{n!}^k))\pi(\delta_k) \in \mathrm{GL}_m(R),$$

where $\mathrm{diag}(\rho(h_1^k), \rho(h_2^k), \ldots, \rho(h_{n!}^k))$ is a block–diagonal matrix; $\pi(\delta_k)$ is a block–monomial matrix in which the block on position $(j, \pi(j))$ is an identical matrix of order l, but the block on position (j, s) for $s \neq \pi(j)$ is a null matrix of order l. This linear representation φ is a faithful representation of $M(0,n)$.

Consider the group $B_n(S^2)$. This group contains the linear subgroup L_n index $2n!$. As a set of right coset representatives of $B_n(S^2)$ by subgroup L_n we take elements $m_i \Delta_n^\epsilon$, $i = 1, 2, \ldots, n!$, $\epsilon = 0, 1$, where Δ_n is the generator of center of $B_n(S^2)$. Since $B_n(S^2)$ is generated by $\delta_1, \delta_2, \ldots, \delta_{n-1}$ then it is enough to define representation φ_1 on these elements. Let us order the coset representatives $m_i \Delta_n^\epsilon$ and denote them by $n_1, n_2, \ldots, n_{2n!}$. Each generator δ_k of $B_n(S^2)$ acts on these representatives by the rules

$$n_j \delta_k = g_j^k n_{\pi_k(j)}, \quad j = 1, 2, \ldots, 2n!, \quad k = 1, 2, \ldots, n-1.$$

We compare δ_k with the matrix

$$\varphi_1(\delta_k) = \mathrm{diag}(\rho(g_1^k), \rho(g_2^k), \ldots, \rho(g_{2n!}^k))\pi(\delta_k) \in \mathrm{GL}_{2m}(R),$$

where $\mathrm{diag}(\rho(g_1^k), \rho(g_2^k), \ldots, \rho(g_{2n!}^k))$ is a block–diagonal matrix and $\pi(\delta_k)$ is a block–monomial matrix in which the block on position $(j, \pi(j))$ is an identical matrix of order l but the block on position (j, s) for $s \neq \pi(j)$ is a null matrix of order l. This linear representation φ_1 is a faithful representation of $B_n(S^2)$ over ring R. Since R is included in the field of complex numbers $\mathbb{C}$ (it is enough to take numbers t and q which are non–zero transcendental over $\mathbb{Q}$) then we obtain the required assertions. □

Let P^2 be a projective plane. The braid group $B_n(P^2)$, $n \geq 1$, admits a presentation [15] with generators $\delta_1, \delta_2, \ldots, \delta_{n-1}$, $\rho_1, \rho_2, \ldots, \rho_n$ and defining relations:

$$\delta_i \delta_{i+1} \delta_i = \delta_{i+1} \delta_i \delta_{i+1} \quad \text{if} \quad i = 1, 2, ..., n-2,$$

$$\delta_i \delta_j = \delta_j \delta_i \quad \text{if} \quad |i - j| \geq 2,$$

$$\delta_i \rho_j = \rho_j \delta_i, \quad j \neq i, i+1,$$

$$\rho_i = \delta_i \rho_{i+1} \delta_i \quad \text{if} \quad i = 1, 2, ..., n-1,$$

$$\rho_{i+1}^{-1} \rho_i^{-1} \rho_{i+1} \rho_i = \delta_i^2 \quad \text{if} \quad i = 1, 2, ..., n-1,$$

$$\delta_1 \delta_2 \ldots \delta_{n-2} \delta_{n-1}^2 \delta_{n-2} \ldots \delta_2 \delta_1 = \rho_1^2.$$

It is clear that the map taking δ_i onto the transposition $(i, i+1)$, $i = 1, 2, \ldots, n-1$ and each ρ_j onto the identity is homomorphism of $B_n(P^2)$ onto the symmetric group S_n, with kernel $P_n(P^2)$.

The group $B_1(P^2)$ is a cyclic group of order 2, $B_2(P^2)$ is a finite group of order 16. For $n \geq 3$ the group $B_n(P^2)$ is infinite. Consider the case $n = 3$. The group $P_3(P^2)$ contains a free subgroup $A_3(P^2) =< \rho_1, a_2, a_3 \parallel a_2 a_3 = \rho_1^2 >$, where $a_2 = \delta_1^2$, $a_3 = \delta_1^{-1}\delta_2^2\delta_1$. The factor group $P_3(P^2)/A_3(P^2) \simeq P_2(P^2)$ is a quaternion group of order 8. The set

$$\{1, \delta_1, \delta_2, \delta_2\delta_1, \delta_1\delta_2, \delta_1\delta_2\delta_1\}$$

is a right coset representatives of $B_3(P^2)$ by $P_3(P^2)$. Note that $P_2(P^2)$ has a presentation

$$P_2(P^2) =< \rho_2, \rho_3 \parallel \rho_2^2 = \rho_3^2 = (\rho_3\rho_2)^2 > .$$

In $P_3(P^2)$ the following equations are true:

$$\rho_3^{-\epsilon} a_2 \rho_3^{\epsilon} = a_2, \quad \epsilon = \pm 1, \quad \rho_j^{-1}\rho_1\rho_j = \rho_1 a_j^{-1}, \ \ j = 2, 3,$$

$$\rho_2^{-1} a_3 \rho_2 = a_2 a_3 a_2^{-1}, \quad \rho_2^{-1} a_2 \rho_2 = \rho_1 a_2^{-1} \rho_1^{-1}.$$

¿From defining relations of $B_3(P^2)$ we get δ_1, δ_2.

Lemma 3.1 *In $B_3(P^2)$ the following rules of conjugating are true*

$$a_2^{\delta_1} = a_2, \quad \rho_1^{\delta_1} = \rho_2 a_2, \quad \rho_2^{\delta_1} = a_2^{-1}\rho_1, \quad \rho_3^{\delta_1} = \rho_3,$$

$$a_2^{\delta_2} = \rho_1^2 a_2^{-1}, \quad \rho_1^{\delta_2} = \rho_1, \quad \rho_2^{\delta_2} = \rho_3\delta_2^2, \quad \rho_3^{\delta_2} = \delta_2^{-2}\rho_2.$$

Since $A_3(P^2)$ is a free group with free generators ρ_1, a_2 then using Sanov's representation [10] we can include $A_3(P^2)$ in $\mathrm{SL}_2(\mathbb{Z})$ assuming

$$\bar{\rho}_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \bar{a}_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

as images of ρ_1 and a_2 accordingly.

The group $P_3(P^2)$ is an extension of $A_3(P^2)$ by the group of quaternion of order 8. Let us chose the set

$$\{e, \rho_2, \rho_2^2, \rho_2^3, \rho_3, \rho_3\rho_2, \rho_3\rho_2^2, \rho_3\rho_2^3\}$$

as the set of coset representatives of $P_3(P^2)$ by $A_3(P^2)$. Thus free group $A_3(P^2)$ has index 48 in $B_3(P^2)$. Using Malcev's construction we get

Theorem 3.2 *Braid group $B_3(P^2)$ of projective plane P^2 is included in* $\mathrm{SL}_{96}(\mathbb{Z})$.

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