EXTENDING REPRESENTATIONS OF BRAID GROUPS TO THE AUTOMORPHISM GROUPS OF FREE GROUPS

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ABSTRACT. We construct a linear representation of the group $\mathrm{IA}(F_n)$ of IA-automorphisms of a free group F_n, an extension of the Gassner representation of the pure braid group P_n. Although the problem of faithfulness of the Gassner representation is still open for $n > 3$, we prove that the restriction of our representation to the group of basis conjugating automorphisms Cb_n contains a non-trivial kernel even if $n = 2$. We construct also an extension of the Burau representation to the group of conjugating automorphisms C_n. This representation is not faithful for $n \geqslant 2$.

One of the generalizations of the braid group B_n on the n strings is the group of conjugating automorphisms C_n. The pure braid group P_n is a normal subgroup of the group B_n. Similarly, the group of basis conjugating automorphisms Cb_n is normal in the group C_n. In both cases, the quotient groups B_n/P_n and C_n/Cb_n are isomorphic to the group S_n, the symmetric group of degree n.

A. G. Savuschkina [14] proved that C_n is a semidirect product: $C_n = Cb_n \leftthreetimes S_n$. For the group B_n the similar statement is not true, since B_n is torsion-free. The group of basis conjugating automorphisms Cb_n is a subgroup of the group $\mathrm{IA}(F_n)$ of the IA-automorphisms of a free group F_n.

Naturally, the solution of the problem of the linearity of the braid groups B_n for all $n \geqslant 2$ [3, 8] initiated the study of the problem of the linearity of C_n (as well as an equivalent problem of linearity of Cb_n) [7, Problem 15.9] and of a more general problem of the linearity of the group $\mathrm{IA}(F_n)$ [7, Problem 15.10].

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One of the possible approaches to solution of these problems is to try to extend known linear representations of P_n to Cb_n (to $\mathrm{IA}(F_n)$) and to try to extend known representations of B_n to C_n. The most famous linear representation of B_n is the Burau representation [5] which is faithful for $n = 3$ and has a non-trivial kernel for all $n > 4$ [13, 9, 2]. In the case when $n = 4$, it is not known whether the Burau representation of B_n is faithful. There is a linear representation of P_n which is called the Gassner representation. The problem of the faithfulness of this representation for $n > 3$ is still open. Construction of both these representations stems from a general construction of the so-called Magnus representation [4, Ch. 3].

In this article using the Magnus representation we will construct a linear representation $\mathrm{IA}(F_n) \longrightarrow \mathrm{GL}_n(\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}])$, which is an extension of the Gassner representation of P_n. Then we can easily obtain an extension of the Burau representation on C_n. Unfortunately, the latter representation is not faithful. Moreover, we will show that its restriction to the group of basis-conjugating automorphisms Cb_n has a non-trivial kernel even if $n = 2$, and so the extension of the Burau representation to C_n is not faithful for $n \geqslant 2$.

As we noted above the Burau representation is not faithful for $n > 4$. However, we apply it to calculate the Alexander polynomials of knots which are closures of the corresponding braids. The Alexander polynomials are one of the most important invariants of the knots. Similarly, C_n is closely related to the so-called welded knots and links, and the linear representations of Cb_n and C_n we have constructed might be used to determine invariants of the welded links.

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1. Preliminaries

The braid group B_n, $n \geqslant 2$, with n strings can be defined as the group generated by $n-1$ elements $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ with the following defining relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ i = 1, 2, ..., n-2, \quad (1)$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \ |i - j| \geqslant 2. \quad (2)$$

There is a homomorphism from the group B_n to the symmetric group S_n of degree n defined via

$$\nu(\sigma_i) = (i, i+1),\ i = 1, 2, \dots, n-1.$$

The kernel of the homomorphism ν is the pure braid group P_n. The group P_n admits a presentation with generators

$$\begin{aligned} &a_{i,i+1} = \sigma_i^2, \\ &a_{ij} = \sigma_{j-1}\sigma_{j-2}\dots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\dots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1},\ i+1 < j \leqslant n, \end{aligned}$$

and the following defining relations:

$$\begin{aligned} &a_{ik}^{-\varepsilon}a_{kj}a_{ik}^{\varepsilon} = (a_{ij}a_{kj})^{\varepsilon}\, a_{kj}\, (a_{ij}a_{kj})^{-\varepsilon}, \\ &a_{km}^{-\varepsilon}a_{kj}a_{km}^{\varepsilon} = (a_{kj}a_{mj})^{\varepsilon}\, a_{kj}\, (a_{kj}a_{mj})^{-\varepsilon}, && m < j, \\ &a_{im}^{-\varepsilon}a_{kj}a_{im}^{\varepsilon} = \left[a_{ij}^{-\varepsilon}, a_{mj}^{-\varepsilon}\right]^{\varepsilon} a_{kj} \left[a_{ij}^{-\varepsilon}, a_{mj}^{-\varepsilon}\right]^{-\varepsilon}, && i < k < m, \\ &a_{im}^{-\varepsilon}a_{kj}a_{im}^{\varepsilon} = a_{kj}, && k < i;\ m < j \text{ or } m < k, \end{aligned}$$

where $\varepsilon = \pm 1$.

The subgroup U_n generated by $a_{1,n}, a_{2,n}, \dots, a_{n-1,n}$ is a free normal subgroup of P_n. The group P_n is a semidirect product of U_n and P_{n-1}. Hence the group P_n is a semi direct product

$$P_n = U_n \leftthreetimes (U_{n-1} \leftthreetimes (\dots \leftthreetimes (U_3 \leftthreetimes U_2))\dots),$$

where $U_i = \langle a_{1,i}, a_{2,i}, \dots, a_{i-1,i}\rangle, i = 2, 3, \dots, n$, is a free group of rank $i-1$.

The braid group B_n can be embedded into the automorphism group of a free group F_n with a free basis $\{x_1, x_2, \dots, x_n\}$. The said embedding is induced by a map from B_n to $\mathrm{Aut}(F_n)$ defined by

$$\sigma_i : \begin{cases} x_i \longmapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \longmapsto x_i, \\ x_l \longmapsto x_l & l \neq i, i+1. \end{cases}$$

The generator a_{rs} of P_n determines the following automorphism of F_n

$$a_{rs} : \begin{cases} x_i \longmapsto x_i & \text{if } \ s < i \text{ or } i < r, \\ x_r \longmapsto x_r x_s x_r x_s^{-1} x_r^{-1}, \\ x_i \longmapsto [x_r^{-1}, x_s^{-1}] x_i [x_r^{-1}, x_s^{-1}]^{-1} & \text{if } \ r < i < s, \\ x_s \longmapsto x_r x_s x_r^{-1}. \end{cases}$$

By a theorem of Artin [4, Theorem 1.9] an automorphism β in $\mathrm{Aut}(F_n)$ belongs to (the image of) B_n if and only if β satisfies the following two conditions

$$\begin{aligned} &1) \quad \beta(x_i) = f_i^{-1} x_{\pi(i)} f_i, \qquad\qquad 1 \leqslant i \leqslant n,\\ &2) \quad \beta(x_1 x_2 \ldots x_n) = x_1 x_2 \ldots x_n, \end{aligned}$$

where π is a permutation of $\{1, 2, \ldots, n\}$ and $f_i = f_i(x_1, x_2, \ldots, x_n)$ is a word in the generators of F_n.

An automorphism of F_n is called a *conjugating automorphism* if it satisfies condition 1). Let C_n be the group of conjugating automorphisms. An automorphism of F_n is called a *basis–conjugating automorphism* if it satisfies condition 1) for the identical permutation π, i. e., maps each generator x_i to a conjugating element. Let Cb_n be the group of basis–conjugating automorphisms. Evidently, Cb_n is normal in C_n and the quotient group C_n/Cb_n is isomorphic to the symmetric group S_n. The elements of the group Cb_n are called *basis-conjugating automorphisms.* J. McCool [12] proved that the group Cb_n is generated by automorphisms

$$\varepsilon_{ij} : \begin{cases} x_i \longmapsto x_j^{-1} x_i x_j, & i \neq j, \\ x_l \longmapsto x_l & l \neq i, \end{cases}$$

$i \leqslant i \neq j \leqslant n.$

Recall (see [10, chapter 1, § 4]) that the group of *IA–automorphisms* $\mathrm{IA}(F_n)$ is generated by the automorphisms ε_{ij}, $1 \leqslant i \neq j \leqslant n$ and the automorphisms

$$\varepsilon_{ijk} : \begin{cases} x_i \longmapsto x_i[x_j, x_k], & \text{if } k \neq i, j, \\ x_l \longmapsto x_l, & \text{if } l \neq i, \end{cases}$$

where $[a, b] = a^{-1} b^{-1} ab$.

2. Fox's derivatives and Magnus representation

Recall the definitions and main properties of Fox's derivatives [4, Chapter 3; 6, Chapter 7].

Let F_n be a free group of rank n with free generators $x_1, x_2, \ldots, x_n$. If φ is any homomorphism defined on F_n, then we use the symbol F_n^{φ} to denote the image of F_n under φ. Consider also the group ring $\mathbb{Z}F_n$ of the group F_n over the ring $\mathbb{Z}$ of integers.

For every $j = 1, 2, \ldots, n$ define the mapping

$$\frac{\partial}{\partial x_j} : \mathbb{Z}F_n \longrightarrow \mathbb{Z}F_n$$

using the following conditions

$$1) \quad \frac{\partial x_i}{\partial x_j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

$$2) \quad \frac{\partial x_i^{-1}}{\partial x_j} = \begin{cases} -x_i^{-1}, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases},$$

$$3) \quad \frac{\partial (wv)}{\partial x_j} = \frac{\partial w}{\partial x_j}(v)^{\tau} + w\frac{\partial v}{\partial x_j}, \; w, v \in \mathbb{Z}F_n,$$

where $\tau : \mathbb{Z}F_n \longrightarrow \mathbb{Z}$ is the operation of trivialization which sends all elements of F_n to 1,

$$4) \quad \frac{\partial}{\partial x_j}\left(\sum a_g g\right) = \sum a_g \frac{\partial g}{\partial x_j}, \; g \in F_n, \; a_g \in \mathbb{Z}.$$

If we denote the fundamental ideal of ring $\mathbb{Z}F_n$ (the kernel of the homomorphism τ) by Δ_n, then it is easy to see that for every $v \in \mathbb{Z}F_n$ the element $v - v^{\tau}$ belongs to Δ_n. The following formula is true:

$$v - v^{\tau} = \sum_{j=1}^{n} \frac{\partial v}{\partial x_j}(x_j - 1);$$

this formula is called the *"Fundamental formula" of free calculus.* In particular, we have as a consequence that $\{x_1 - 1, x_2 - 1, \ldots, x_n - 1\}$ is a basis of the fundamental ideal Δ_n.

We shall need Blanchfield's theorem [4, Theorem 3.5] which says that if φ be an arbitrary homomorphism acting on F_n, an element $v \in F_n$ lies in the commutator subgroup $[\ker\varphi, \ker\varphi]$ if and only if $\left(\frac{\partial v}{\partial x_j}\right)^{\varphi} = 0$ for all $j = 1, 2, \ldots, n$.

Let A_{φ} be any subgroup of $\mathrm{Aut}F_n$ which satisfies the condition

$$x^{\varphi} = (x^{\alpha})^{\varphi}$$

for every $x \in F_n$ and for every $\alpha \in A_{\varphi}$. If $\alpha \in A_{\varphi}$, we define $||\alpha||$ to be the $n \times n$ matrix

$$||\alpha|| = \left[\left(\frac{\partial (x_i^{\alpha})}{\partial x_j}\right)^{\varphi}\right]_{i,j=1}^{n}$$

with entries in $\mathbb{Z}F_n^\varphi$. This mapping defines the Magnus representation

$$\rho : A_\varphi \longrightarrow \mathrm{GL}_n(\mathbb{Z}F_n^\varphi).$$

Taking as φ a homomorphism from F_n onto an infinite cyclic group $\langle t \rangle$, that is, assuming that

$$x_i^\varphi = t, \quad i = 1, 2, \ldots, n,$$

we obtain the Burau representation

$$\rho_B : B_n \longrightarrow \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}])$$

of the braid group B_n.

Similarly, if φ is a homomorphism from F_n onto a free abelian group A_n with free generators $t_1, t_2, \ldots, t_n$,

$$x_i^\varphi = t_i, \quad i = 1, 2, \ldots, n$$

then the resulting representation is the Gassner representation

$$\rho_G : P_n \longrightarrow \mathrm{GL}_n(\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}])$$

of the group P_n.

3. The construction of $\widehat{\rho}_G$

It is easy to check that for every $x \in F_n$ and every automorphism $\alpha \in \mathrm{IA}(F_n)$ the following equality is true:

$$x^\varphi = (x^\alpha)^\varphi,$$

where the homomorphism $\varphi : F_n \longrightarrow A_n$ is defined in the end of the previous section. Consequently, we can construct the Magnus representation

$$\rho : \mathrm{IA}(F_n) \longrightarrow \mathrm{GL}_n(R),$$

where $R = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$. In order to define the action of ρ on generators of $\mathrm{IA}(F_n)$ we should calculate the Fox's derivations.

Lemma 1. *The following formulas are true in the group ring* $\mathbb{Z}F_n$ (*where* i, j, k *are pairwise distinct*):

$$\frac{\partial x_l^{\varepsilon_{ijk}}}{\partial x_m} = 0, \qquad m \notin \{l, i, j, k\},$$

$$\frac{\partial x_l^{\varepsilon_{ijk}}}{\partial x_l} = 1, \qquad l \notin \{j, k\},$$

$$\frac{\partial x_i^{\varepsilon_{ijk}}}{\partial x_j} = -x_i x_j^{-1} + x_i x_j^{-1} x_k^{-1},$$

$$\frac{\partial x_i^{\varepsilon_{ijk}}}{\partial x_k} = -x_i x_j^{-1} x_k^{-1} + x_i x_j^{-1} x_k^{-1} x_j.$$

We will consider the matrix $\rho(\varepsilon_{ijk})$ as an automorphism of a free left R–module W_n with base $e_1, e_2, \ldots, e_n$.

Then the action of this automorphism (from the right) on the base vectors is as follows:

$$\begin{cases} e_i\rho(\varepsilon_{ijk}) = e_i + t_i t_j^{-1}(t_k^{-1} - 1)e_j + t_i t_k^{-1}(1 - t_j^{-1})e_k, & \\ e_l\rho(\varepsilon_{ijk}) = e_l & \text{if } l \neq i. \end{cases}$$

In order to construct the matrices $\rho(\varepsilon_{ij})$, we will use the following simple lemma.

Lemma 2. *The following formulas are true in the group ring* $\mathbb{Z}F_n$ (*where* i, j, l, m *are pairwise distinct*):

$$\frac{\partial x_l^{\varepsilon_{ij}}}{\partial x_m} = 0, \ \frac{\partial x_l^{\varepsilon_{ij}}}{\partial x_l} = 1, \ \frac{\partial x_i^{\varepsilon_{ij}}}{\partial x_i} = x_j^{-1}, \ \frac{\partial x_i^{\varepsilon_{ij}}}{\partial x_j} = x_j^{-1}(x_i - 1).$$

$$\frac{\partial x_l^{\varepsilon_{ij}^{-1}}}{\partial x_m} = 0, \ \frac{\partial x_l^{\varepsilon_{ij}^{-1}}}{\partial x_l} = 1, \ \frac{\partial x_i^{\varepsilon_{ij}^{-1}}}{\partial x_i} = x_j, \ \frac{\partial x_i^{\varepsilon_{ij}^{-1}}}{\partial x_j} = 1 - x_j x_i x_j^{-1}.$$

Then the matrix $\rho(\varepsilon_{ij})$ defines the action on the base of W_n by the next formulas:

$$\begin{cases} e_i\rho(\varepsilon_{ij}) = t_j^{-1}(t_i - 1)e_j + t_j^{-1}e_i, & \\ e_l\rho(\varepsilon_{ij}) = e_l & \text{if } l \neq i. \end{cases}$$

For the automorphism

$$\varepsilon_{ij}^{-1} : \begin{cases} x_i \longmapsto x_j x_i x_j^{-1} & \text{if } i \neq j, \\ x_l \longmapsto x_l & \text{if } l \neq i, \end{cases}$$

the matrix $\rho(\varepsilon_{ij}^{-1})$ is defined by the action on the base of W_n as follows:

$$\begin{cases} e_i \rho(\varepsilon_{ij}^{-1}) = t_j e_i + (1 - t_i) e_j, \\ e_l \rho(\varepsilon_{ij}^{-1}) = e_l & \text{if } l \neq i. \end{cases}$$

It is easy to check that being defined in this way the mapping ρ is a linear representation of $\mathrm{IA}(F_n)$.

Restricting ρ to Cb_n, we obtain that

$$\widehat{\rho}_G = \rho|_{Cb_n} : Cb_n \longrightarrow \mathrm{GL}_n(R).$$

Theorem 1. *The linear representation*

$$\widehat{\rho}_G : Cb_n \longrightarrow \mathrm{GL}_n(R)$$

is an extension of the Gassner representation of the pure braid group P_n.

Proof. The group P_n is a subgroup of Cb_n and its generators a_{ij}, $1 \leqslant i < j \leqslant n$ can be expressed via the (standard) generators of Cb_n in the following way [1, Lemma 4]:

$$\begin{aligned} a_{i,i+1} &= \varepsilon_{i,i+1}^{-1} \varepsilon_{i+1,i}^{-1}, & i = 1, 2, \ldots, n-1, \\ a_{ij} &= \varepsilon_{j-1,i} \varepsilon_{j-2,i} \ldots \varepsilon_{i+1,i} (\varepsilon_{ij}^{-1} \varepsilon_{ji}^{-1}) \varepsilon_{i+1,i}^{-1} \ldots \varepsilon_{j-2,i}^{-1} \varepsilon_{j-1,i}^{-1} & \\ &= \varepsilon_{j-1,j}^{-1} \varepsilon_{j-2,j}^{-1} \ldots \varepsilon_{i+1,j}^{-1} (\varepsilon_{ij}^{-1} \varepsilon_{ji}^{-1}) \varepsilon_{i+1,j} \ldots \varepsilon_{j-2,j} \varepsilon_{j-1,j}, & 2 \leqslant i+1 < j \leqslant n. \end{aligned}$$

Using these formulas we find the matrix $\rho(a_{ij})$, $1 \leqslant i < j \leqslant n$ and then comparing it with the Gassner matrix from [4, p. 118] we get the desired conclusion.

Recall that the group of conjugating automorphisms C_n can be decomposed as a semidirect product $C_n = Cb_n \leftthreetimes S_n$.

As the matrix $\widehat{\rho}_G(\varepsilon_{ij})$ depends on $t_1, t_2, \ldots, t_n$, we may set

$$t_1 = t_2 = \cdots = t_n = t,$$

thereby obtaining the matrix which we shall denote by $\widehat{\rho}_B(\varepsilon_{ij})$. Further, for each automorphism from $S_n \leqslant \mathrm{Aut}(F_n)$ we assign the matrix of the corresponding permutation of the elements of the base W_n. We will then obtain the representation

$$\widehat{\rho}_B : C_n \longrightarrow \mathrm{GL}_n(\mathbb{Z}[t^{\pm}]).$$

It is easy to check that $\widehat{\rho}$ is an extension of the Burau representation of the braid group B_n.

As Cb_n is a subgroup of finite index in C_n, the representation $\widehat{\rho}_B$ is faithful if its restriction to Cb_n is faithful.

It is well-known that the Burau representation ρ_B and the Gassner representations ρ_G are reducible; however, both these representations determine some irreducible representations of dimension $n-1$ [4, Lemma 3.11.1].

The following question naturally arises.

Question. Is it true that the presentations $\widehat{\rho}_B$ and $\widehat{\rho}_G$ are reducible?

4. Kernel of the representation $\widehat{\rho}_G$

In this section we will show that the representation $\widehat{\rho}_G$ is not faithful. We shall prove the following result.

Theorem 2. *The representation $\widehat{\rho}_G : Cb_n \longrightarrow \mathrm{GL}_n(R)$ has a non-trivial kernel for every $n \geqslant 2$.*

The theorem implies that the representation $\rho : \mathrm{IA}(F_n) \longrightarrow \mathrm{GL}_n(R)$ has a non-trivial kernel for every $n \geqslant 2$.

Recall [1] that the group of basis–conjugating automorphisms Cb_n, $n \geqslant 2$ can be decomposed as a semidirect product as follows:

$$Cb_n = D_{n-1} \leftthreetimes (D_{n-2} \leftthreetimes (\ldots \leftthreetimes (D_2 \leftthreetimes D_1)) \ldots),$$

where subgroup D_i is generated by the elements

$$\varepsilon_{i+1,1}, \varepsilon_{i+1,2}, \ldots, \varepsilon_{i+1,i}, \varepsilon_{1,i+1}, \varepsilon_{2,i+1}, \ldots, \varepsilon_{i,i+1}.$$

Moreover, the elements $\varepsilon_{i+1,1}, \varepsilon_{i+1,2}, \ldots, \varepsilon_{i+1,i}$ generate a free group of rank i which we will denote by L_i and the elements $\varepsilon_{1,i+1}, \varepsilon_{2,i+1}, \ldots, \varepsilon_{i,i+1}$ generate a free abelian group of rank i which we will denote by A_i.

Let us show that if $j \geqslant 3$ then the second commutator subgroup L_j'' is contained in the kernel of the representation $\widehat{\rho}_G$. Following the construction of the matrix $\widehat{\rho}_G(\varepsilon_{ij}) = \rho(\varepsilon_{ij})$, one sees that it different from the identity matrix only in ith row. Let $w = w(\varepsilon_{i1}, \varepsilon_{i2}, \ldots, \varepsilon_{i,i-1})$ be a reduced word over the elements $\varepsilon_{i1}^{\pm 1}, \varepsilon_{i2}^{\pm 1}, \ldots, \varepsilon_{i,i-1}^{\pm 1}$, the free generators of L_{i-1}. The following lemma describes how the automorphism w acts on a generator x_i of the free group F_n.

Lemma 3. *Let* $w = w(\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{i,i-1})$ *be a reduced word which represents an element of* L_{i-1}. *Then*

$$x_i w(\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{i,i-1}) = x_i^{w^*(x_1, x_2, \dots, x_{i-1})},$$

where the word $w^*(x_1, x_2, \dots, x_{i-1})$ *is the* reverse *word of the word* $w(x_1, x_2, \dots, x_{i-1})$, *that is the syllables of* w^* *are the syllables of* w *written in the reverse order.*

Proof. The statement is a consequence of the following equations that can be easily verified by induction on the number of syllables of w (the *syllabic length* of w):

$$x_i \varepsilon_{ik}^p = x_i^{x_k^p}, \ x_i \varepsilon_{ik}^p \varepsilon_{il}^q = x_i^{x_l^q x_k^p}, \ 1 \leqslant k \neq l \leqslant i-1, \ p, q \in \mathbb{Z},$$

Obviously, if $w = w(\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{i,i-1})$ represents an element of the second commutator subgroup L''_{i-1}, then the word $w^* = w^*(x_1, x_2, \dots, x_{i-1})$ represents an element of the second commutator subgroup F''_n.

Assume that $w = w(\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{i,i-1})$ represents an element of L''_{i-1}. In order to find the matrix $\widehat{\rho}_G(w)$ we have to find the derivatives $\dfrac{\partial(x_i w)}{\partial x_k}$, $k = 1, 2, \dots, n$. Let us check the following equality:

$$\left(\frac{\partial(x_i w)}{\partial x_k}\right)^{\varphi} = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{otherwise.} \end{cases}$$

The case when $k > i$ and the case when the word $x_i w$ does not contain x_k are simple. Suppose that $k = i$. We have

$$\frac{\partial(x_i w)}{\partial x_i} = \frac{\partial((w^*)^{-1} x_i w^*)}{\partial x_i}.$$

Using properties of the Fox's derivatives, we obtain that

$$\begin{aligned}\frac{\partial((w^*)^{-1} x_i w^*)}{\partial x_i} &= \frac{\partial((w^*)^{-1})}{\partial x_i} + (w^*)^{-1} \frac{\partial(x_i w^*)}{\partial x_i} \\ &= -(w^*)^{-1} \frac{\partial w^*}{\partial x_i} + (w^*)^{-1} \left(1 + x_i \frac{\partial w^*}{\partial x_i}\right).\end{aligned}$$

As w^* does not contain x_i, the derivatives $\dfrac{\partial w^*}{\partial x_i}$ become zero and w^* lies in the commutator subgroup F'_n and φ takes it to 1, that is, $(w^*)^{\varphi} = 1$, as required.

Let, finally, $k < i$. Then

$$\frac{\partial((w^*)^{-1}x_i w^*)}{\partial x_k} = (w^*)^{-1}(x_i - 1)\frac{\partial w^*}{\partial x_k}.$$

Since $\ker\varphi = F_n'$, then by Blanchfield's theorem (taking into account that $w^* \in F_n''$) we have the equality $\left(\frac{\partial w^*}{\partial x_k}\right)^{\varphi} = 0$. The statement is proven.

As a consequence we have the following fact.

Lemma 4. *The representation $\widehat{\rho}_G$ of Cb_n, $n \geqslant 3$, is not faithful, since its kernel contains the subgroups L_i'', $i = 2, 3, \ldots, n-1$. In particular, the above constructed representation $\widehat{\rho}_B$ of C_n, an extension of the Burau representation, is not faithful for all $n \geqslant 3$.*

Let us demonstrate that actually the representation $\widehat{\rho}_G$ is not faithful even if $n = 2$. In order to prove that, we shall need

Lemma 5. *The following formulas*

$$x_i w(\varepsilon_{12}, \varepsilon_{21}) = x_i^{w(x_2, x_1)}, \ \ i = 1, 2.$$

are true in $F_2 = \langle x_1, x_2\rangle$. In particular, if $w(\varepsilon_{12}, \varepsilon_{21})$ is in the second commutator subgroup D_1'', then $w(x_2, x_1)$ is in the second commutator subgroup F_2''.

Proof. Recall that $Cb_2 = D_1 = \langle \varepsilon_{21}, \varepsilon_{12}\rangle \simeq F_2$ and the automorphisms ε_{12} and ε_{21} are defined as follows:

$$\varepsilon_{12} : \begin{cases} x_1 \longmapsto x_2^{-1}x_1x_2, \\ x_2 \longmapsto x_2, \end{cases} \qquad \varepsilon_{21}^{-1} : \begin{cases} x_1 \longmapsto x_1, \\ x_2 \longmapsto x_1^{-1}x_2x_1, \end{cases}$$

i. e. ε_{12} induces conjugation by x_2 in F_2 and ε_{21} conjugation by x_1. The action of $\varepsilon_{12}\varepsilon_{21}$ on the base x_1, x_2 is

$$\varepsilon_{12}\varepsilon_{21} : \begin{cases} x_1 \longmapsto x_1^{-1}x_2^{-1}x_1x_2x_1, \\ x_2 \longmapsto x_1^{-1}x_2x_1, \end{cases}$$

which means that $x_i\varepsilon_{12}\varepsilon_{21} = x_i^{x_2x_1}$, $i = 1, 2$.

If $w = w(\varepsilon_{12}, \varepsilon_{21})$ is a reduced word over the alphabet $\{\varepsilon_{12}^{\pm 1}, \varepsilon_{21}^{\pm 1}\}$, then using induction on the length of w, we get the following statement.

Lemma 6. *The kernel of the representation $\widehat{\rho}_G$ of Cb_2 coincides the second commutator subgroup D_1''.*

Proof. Let us show that if $w = w(\varepsilon_{12}, \varepsilon_{21})$ is in D_1'' then $\widehat{\rho}_G(w) = 1$. It follows from the definition of the Magnus representation that

$$\widehat{\rho}_G(w(\varepsilon_{12}, \varepsilon_{21})) = \begin{pmatrix} \dfrac{\partial(x_1 w(\varepsilon_{12}, \varepsilon_{21}))}{\partial x_1} & \dfrac{\partial(x_1 w(\varepsilon_{12}, \varepsilon_{21}))}{\partial x_2} \\ \dfrac{\partial(x_2 w(\varepsilon_{12}, \varepsilon_{21}))}{\partial x_1} & \dfrac{\partial(x_2 w(\varepsilon_{12}, \varepsilon_{21}))}{\partial x_2} \end{pmatrix}^{\varphi}.$$

Write w_1 for $w(x_2, x_1)$. We calculate the Fox's derivatives:

$$\frac{\partial(x_1 w)}{\partial x_1} = \frac{\partial(w_1^{-1} x_1 w_1)}{\partial x_1} = \frac{\partial(w_1^{-1})}{\partial x_1} + w_1^{-1}\frac{\partial(x_1 w_1)}{\partial x_1} =$$

$$= -w_1^{-1}\frac{\partial w_1}{\partial x_1} + w_1^{-1}\left(1 + x_1\frac{\partial w_1}{\partial x_1}\right) = w_1^{-1}\left(-\frac{\partial w_1}{\partial x_1} + 1 + x_1\frac{\partial w_1}{\partial x_1}\right) =$$

$$= w_1^{-1}\left(1 + (x_1 - 1)\frac{\partial w_1}{\partial x_1}\right).$$

Using the homomorphism φ and taking into account the following formulas

$$(w_1^{-1})^{\varphi} = 1, \ \left(\frac{\partial w_1}{\partial x_1}\right)^{\varphi} = 0,$$

we obtain that

$$\left(\frac{\partial(x_1 w)}{\partial x_1}\right)^{\varphi} = 1.$$

Then

$$\frac{\partial(x_1 w)}{\partial x_2} = \frac{\partial(w_1^{-1} x_1 w_1)}{\partial x_2} = \frac{\partial(w_1^{-1})}{\partial x_2} + w_1^{-1}\frac{\partial(x_1 w_1)}{\partial x_2} =$$

$$= -w_1^{-1}\frac{\partial w_1}{\partial x_2} + w_1^{-1} x_1 \frac{\partial w_1}{\partial x_2} = -w_1^{-1}(1 - x_1)\frac{\partial w_1}{\partial x_1}.$$

Since $\left(\dfrac{\partial w_1}{\partial x_2}\right)^{\varphi} = 0$, then $\left(\dfrac{\partial(x_1 w)}{\partial x_2}\right)^{\varphi} = 0$. Similarly,

$$\frac{\partial(x_2 w)}{\partial x_1} = -w_1^{-1}(1 - x_2)\frac{\partial w_1}{\partial x_1}, \ \frac{\partial(x_2 w)}{\partial x_2} = w_1^{-1}\left(1 + (x_2 - 1)\frac{\partial w_1}{\partial x_2}\right).$$

Using the homomorphism φ we see that

$$\widehat{\rho}_G(w(\varepsilon_{12}, \varepsilon_{21})) = 1.$$

The proof is completed.

Now Lemma 4 and Lemma 6 imply Theorem 2. It then follows that the representations $\rho : \mathrm{IA}(F_n) \longrightarrow \mathrm{GL}_n(R)$ and $\widehat{\rho}_B : C_n \longrightarrow \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}])$ are not faithful for all $n \geqslant 2$. Note that the question about faithfulness of the Gassner representation of P_n, $n \geqslant 4$, is still open.

It is interesting to note that Cb_2 is generated by elements ε_{12} and ε_{21} and corresponding matrices $\widehat{\rho}_G(\varepsilon_{12}^{-1})$ and $\widehat{\rho}_G(\varepsilon_{21}^{-1})$ are

$$\widehat{\rho}_G(\varepsilon_{12}^{-1}) = \begin{pmatrix} \dfrac{\partial(x_1\varepsilon_{12}^{-1})}{\partial x_1} & \dfrac{\partial(x_1\varepsilon_{12}^{-1})}{\partial x_2} \\ \\ \dfrac{\partial(x_2\varepsilon_{12}^{-1})}{\partial x_1} & \dfrac{\partial(x_2\varepsilon_{12}^{-1})}{\partial x_2} \end{pmatrix}^{\varphi} = \begin{pmatrix} t_2 & 1-t_1 \\ 0 & 1 \end{pmatrix},$$

$$\widehat{\rho}_G(\varepsilon_{21}^{-1}) = \begin{pmatrix} \dfrac{\partial(x_1\varepsilon_{21}^{-1})}{\partial x_1} & \dfrac{\partial(x_1\varepsilon_{21}^{-1})}{\partial x_2} \\ \\ \dfrac{\partial(x_2\varepsilon_{21}^{-1})}{\partial x_1} & \dfrac{\partial(x_2\varepsilon_{21}^{-1})}{\partial x_2} \end{pmatrix}^{\varphi} = \begin{pmatrix} 1 & 0 \\ 1-t_2 & t_1 \end{pmatrix}.$$

As we saw in the proof of the last Lemma these matrices do not generate a free group. But the matrices

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}, \ \lambda, \mu \in \mathbb{C}, \ |\lambda| \geqslant 3, |\mu| \geqslant 3,$$

generate a group which is isomorphic to the free group F_2.

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