Limit Theorems in Probability Theory and Their Applications
15 - 21 August
Novosibirsk - 2011

Random walks in cones

Denis Denisov[1] and Vitali Wachtel[2]

We consider a random walk $S_n = (S_n^{(1)}, \dots, S_n^{(d)})$ on $\mathbf{R}^d$, where

$$S_n^{(j)} = \xi_1^{(j)} + \dots + \xi_n^{(j)}, \quad j = 1, \dots, d,$$

and $\{\xi_n, n \geq 1\}$ is a family of independent and identically distributed random vectors. Let K be a cone generated by the rays eminating from the origin. Let τ_x be the exit time from K of the random walk with starting point $x \in K$, that is,

$$\tau_x = \inf\{n \geq 1 : x + S_n \notin K\}.$$

We study the asymptotic behaviour of the random walk S_n conditioned to stay in the cone K. We define a conditional version of a random walk via Doob $h-$transform. For that we construct a function h (which is usually called *invariant function*) such that $h(x) > 0$ for all $x \in K$ and

$$\mathbf{E}[h(x + S(1)); \tau_x > 1] = h(x), \quad x \in K. \tag{1}$$

Then one can make a *change of measure* $\widehat{\mathbf{P}}_x^{(h)}(S_n \in dy) = \mathbf{P}(x + S_n \in dy, \tau_x > n)\frac{h(y)}{h(x)}$. As a result, one obtains a Markov S_n under a new measure $\widehat{\mathbf{P}}_x^{(h)}$ which lives on the state space K.

In the present paper we construct a function $V(x)$ satisfying (1). The construction of the function V is closely connected with the asymptotic behaviour of $\mathbf{P}(\tau_x > n)$. The invariant function reflects the dependence of $\mathbf{P}(\tau_x > n)$ on the starting point x.

We impose the following assumptions on the increments of random walk

- *Centering assumption:* We assume that $\mathbf{E}\xi^{(j)} = 0, \mathbf{E}(\xi^{(j)})^2 = 1, j = 1, \dots, d$. In addition we assume that $cov(\xi^{(i)}, \xi^{(j)}) = 0$.

- *Moment assumption:* We assume that $\mathbf{E}|\xi|^\alpha < \infty$ with $\alpha = p$ if $p > 2$ and some $\alpha > 2$ if $p \leq 2$.

Here is our *main* result:

Theorem 1. *Assume the centering as well as the moment assumption hold. Then there exists a finite and strictly positive function V satisfying (1). Moreover, as $n \to \infty$,*

$$\mathbf{P}(\tau_x > n) \sim \varkappa V(x) n^{-p/2}, \quad x \in K, \tag{2}$$

where $\varkappa$ is an absolute constant. The conditional random walk $\widehat{S}_n$ can be well defined using Doob h-transform with $h = V$.

Then we turn to the asymptotic behaviour of $\widehat{S}_n$. Under the above assumptions we show that the process $\widehat{S}_{nt}$ converges to the corresponding Brownian motion conditioned to stay in the cone. Finally we determine the asymptotic behaviour of local probabilities for random walks conditioned to stay in a cone.

---

[1]School of Mathematics, Cardiff University, Senghennydd Road CARDIFF, Wales, UK. CF24 4AG Cardiff. E-mail: DenisovD@cf.ac.uk

[2]Mathematical Institute, University of Munich, Theresienstrasse 39, D–80333 Munich, Germany. E-mail: wachtel@mathematik.uni-muenchen.de