On Limit Theorems for generalized sums

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Let $\{\xi_n,\ n = 1, 2, \dots\}$ be a sequence of i.i.d. random variables and $x \oplus y$ — binary operation on $\mathbb{D} \subseteq \mathbb{R}$, satisfying the following conditions:

A_1. associativity: $x \oplus (y \oplus z) = (x \oplus y) \oplus z, \quad x,\ y,\ z \in \mathbb{D}$;

A_2. commutativity: $x \oplus y = y \oplus x, \quad x,\ y \in \mathbb{D}$;

A_3. $x \oplus 0 = x,\ \ x \in \mathbb{D}$;

A_4. uniform continuity in the following sense: for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|y| < \delta$ implies $|x \oplus y - x| < \varepsilon,\ \ \forall x \in \mathbb{D}$.

Conditions $A_1 - A_4$ are fulfilled, e.g. for $x \oplus y = x + y$, $x \oplus y = \max\{x,\ y\}$, $x \oplus y = \sqrt{x^2 + y^2}$, $\mathbb{D} = \mathbb{R}_+ = [0, +\infty)$, and so on, and not fulfilled, say, for $x \oplus y = xy$, $x \oplus y = x + y\ (mod\ d)$, $d > 0$ (does not satisfied A_4). Denote $X_n(b) = (\xi_1/b) \oplus ... \oplus (\xi_n/b)$, $b > 0$.

Theorem 1. *If a sequence of positive numbers $\{c_n\}$ is such that for any $\delta > 0$*

$$\max_{1 \le k \le n} \mathbb{P}\{X_k(c_n) \ge \delta\} \to 0,\ \ n \to \infty,$$

and for all $x > 0,\ \ \delta > 0$

$$\mathbb{P}\{|X_n(c_n)| \ge \delta\} = O\left(\mathbb{P}\{|X_n(c_n)| \ge x\}\right),$$

then

$$\mathbb{P}\{|X_n(c_n)| \ge x\} \sim n\mathbb{P}\{\xi_1 \ge xc_n\},\ \ n \to \infty.$$

Theorem 1 means, that under the assumptions above asymptotics of the tails distributions of variables $X_n(c_n)$ does not depend on the type of operation $\oplus$. This result allows us to prove limit theorems as the following Theorem 2.

Let for any $n \in \mathbb{N}$ and $z > 0$ $\mathbb{E}|X_n(z)|^p < \infty$, and

$$b_n(p) = \inf\left\{z > 0 : \max_{1 \le k \le n} \mathbb{E}|X_k(z)|^p \le 1\right\}.$$

In the proof of limit theorems we will use the following conditions

$$\liminf_{n \to \infty} n\mathbb{P}\{\xi_1 \ge \varepsilon b_n(p)\} > 0, \quad \text{for some } \varepsilon > 0; \qquad (1)$$

A_5. For any $x > 0,\ \ y_i > 0,\ \ n \in \mathbb{N}\ \ (xy_1) \oplus ... \oplus (xy_n) = x(y_1 \oplus ... \oplus y_n)$.

Theorem 2. *Assume that the operation $\oplus$ satisfies $A_1 - A_5$ on $\mathbb{D} = [0, \infty)$ and for any $x > 0$ there exists a limit $1 - G(x) = \lim_{k \to \infty} \left(1 - \dfrac{x^{-\rho}}{k}\right)_+^{*k}$, where $F_\xi * F_\eta = F_{\xi \oplus \eta}$, F_ξ - d.f. of ξ, $a_+ = \max(a, 0)$, and, besides, holds (1).*

In order to exist regularly varying with the exponent $-\rho,\ \ \rho > 0$, limit

$$\lim_{n \to \infty} \mathbb{P}\{X_n(a_n) \ge x\} = G(x), \quad (a_n = \sup\{x > 0 : n\mathbb{P}\{\xi_1 \ge x\} \ge 1\})$$

it is necessary and sufficient, that $\mathbb{P}\{\xi_1 \ge x\}$ is regularly varying function with the exponent $-\rho$.