Small deviations of Brownian functionals: exact asymptotics

Ya. Yu. Nikitin, R. S. Pusev

Saint-Petersburg State University
Saint-Petersburg, Russia

We find exact small deviation asymptotics for some Brownian functionals beginning by the Wiener process W on $[0,1]$ in the weighted L_2-norm $\|\cdot\|_\psi$ for a large class of weights ψ.

Theorem 1. *Let the weight ψ, defined on $[0,1]$, be positive and twice continuously differentiable. Put $\vartheta = \int_0^1 \sqrt{\psi(t)}dt$. Then as $\varepsilon \to 0$*

$$\mathsf{P}\{\|W\|_\psi \leqslant \varepsilon\} \sim \frac{4\psi^{1/8}(0)}{\sqrt{\pi}\vartheta\psi^{1/8}(1)} \varepsilon \exp\left(-\frac{\vartheta^2}{8}\varepsilon^{-2}\right). \tag{1}$$

Similar theorems are proved for Brownian bridge, Ornstein-Uhlenbeck process and some similar Gaussian processes. From this we deduce many exact small deviation results for integral functionals of weighted Bessel processes and bridges, Brownian local times, and related processes. Next theorem gives the exact small deviation asymptotics for the Brownian excursion $\mathfrak{e}$ and the Brownian meander $\mathfrak{m}$ in the weighted quadratic norm.

Theorem 2. *Under conditions of Theorem 1, one has as $\varepsilon \to 0$*

$$\mathsf{P}\{\|\mathfrak{e}\|_\psi \leqslant \varepsilon\} \sim \frac{2\sqrt{6\vartheta}\psi^{3/8}(0)\psi^{3/8}(1)}{\sqrt{\pi}} \varepsilon^{-2} \exp\left(-\frac{9\vartheta^2}{8}\varepsilon^{-2}\right),$$

$$\mathsf{P}\{\|\mathfrak{m}\|_\psi \leqslant \varepsilon\} \sim 4\sqrt{\frac{2}{3\pi}} \frac{\psi^{3/8}(0)}{\vartheta^{1/2}\psi^{1/8}(1)} \exp\left(-\frac{9\vartheta^2}{8}\varepsilon^{-2}\right).$$

The relations of Theorem 2 are new even for the unit weight $\psi \equiv 1$.

Let $L_t^x(B)$ be the jointly continuous local time of a Brownian bridge B at the point $x \in \mathbb{R}$ up to time $t \in [0,1]$.

Theorem 3. *The following relation holds as $\varepsilon \to 0$:*

$$\mathsf{P}\left\{\int_{-\infty}^{\infty} (L_1^x(B))^3 dx \leqslant \varepsilon\right\} \sim \frac{8\sqrt{6}}{\sqrt{\pi}} \varepsilon^{-1} \exp\left(-\frac{9}{2}\varepsilon^{-1}\right).$$

This proposition refines on the result from [1], where the asymptotic relation was proved at the logarithmic level only. Many analogous results for other Brownian functionals can be found in [2].

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[1] Csörgő, M., Shi, Z. and Yor, M. (1999). Some asymptotic properties of the local time of the uniform empirical process. *Bernoulli* **5** 1035–1058.

[2] Nikitin, Ya. Yu. and Pusev, R. S. (2011). Exact L_2-small deviation asymptotics for some Brownian functionals. Available at http://arxiv.org/abs/1104.2891