On unimprovable estimates of accuracy for asymptotic expansions in CLT

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In the report asymptotic expansions in the CLT for smooth distribution are considered. For some expansions one of which is formulated below, estimates of residual parts are unimprovable.

Let X_1, X_2, ... be independent identically distributed random variables with zero means, unit variances, third moments α_3 and fourth moments α_4. Refer to the distribution of these random variables as smooth whenever its characteristic function f satisfies $\int\limits_{-\infty}^{\infty} |f(t)|^{\nu}\,dt < \infty$ for some $\nu > 0$. In this case the value $\alpha(T) = \max\{|f(t)| : t \geqslant T\} < 1$ for all $T > 0$, and for any $n \geqslant \nu$ there exist the density $p_n(x)$ of distribution of normalized sum $(X_1 + ... + X_n)/\sqrt{n}$. Denote $B_{k,n} = \frac{1}{2\pi} \int\limits_{-T\sqrt{n}}^{T\sqrt{n}} |t|^k \mu^{n-1}(t/\sqrt{n})\,dt$, where $\mu(t) = \max\left\{|f(t)|,\, e^{-t^2/2}\right\}$ and $T > 0$. At our restrictions there exists $T > 0$ such that for any $k > 0$ and $n \to \infty$

$$B_{k,n} \to B_k = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} |x|^k \varphi(x)\,dx,$$

where $\varphi(x)$ is the density of the standard normal distribution.

For any $n \geqslant \nu$ and $T > 0$

$$p_n(x) = \varphi(x) + \frac{\alpha_3}{6\sqrt{n}} H_3(x)\varphi(x) - \frac{1}{8n} H_4(x)\varphi(x) + \frac{\alpha_3^2}{72n} H_6(x)\varphi(x) + R,$$

where $H_k(x) = (-1)^k \varphi^{(k)}(x)/\varphi(x)$ are Chebyshev - Hermite polynomials,

$$|R| \leqslant \frac{\alpha_4}{24n} B_{4,n} + \frac{|\alpha_3|}{12n^{3/2}} B_{5,n} + \frac{|\alpha_3|}{48n^{3/2}} B_7 + \frac{|\alpha_3|^3}{1296n^{3/2}} B_9 +$$
$$+ \frac{\alpha_3^2 + 3}{72n^2} B_{6,n} + \frac{\alpha_4 + 3}{384n^2} B_{8,n} + \frac{|\alpha_3|}{192n^{5/2}} B_{9,n} + \frac{\alpha_3^2}{1152n^3} B_{10,n} +$$
$$+ \frac{\sqrt{n}}{\pi} \alpha^{n-\nu}(T) \int\limits_{T}^{\infty} |f(t)|^{\nu}\,dt + \frac{1}{8\pi n} \int\limits_{T\sqrt{n}}^{\infty} t^4 e^{-t^2/2} dt + \frac{1}{\pi T\sqrt{n}} e^{-T^2 n/2}.$$

This estimate of the value $|R|$ is unimprovable in the sense that its validity is violated if one multiply the first summand in its right part to any constant c, $0 < c < 1$.