# The probabilistic representation of the exponent of a class of pseudo-differential operators

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We consider an evolution equation $\frac{\partial u}{\partial t} = \mathcal{A}u$, where $\mathcal{A}$ is a linear operator. Given an operator $\mathcal{A}$ let $e^{t\mathcal{A}},\ t \geq 0$ denote the operator exponent that is a family of linear operators such that for every $t > 0$ the operator $e^{t\mathcal{A}}$ maps the real or complex-valued function $\varphi(x), x \in \mathbb{R}$ into a solution $u(t,x)$ of the Cauchy problem of the equation with an initial condition $u(0,x) = \varphi(x)$.

It is well known, that the exponents of some pseudo-differential operators and the exponent of the operator $\frac{d^2}{dx^2}$ have probabilistic representations. Namely, for the operator $\mathcal{A} = \frac{d^2}{dx^2}$ we have $e^{t\mathcal{A}}\varphi(x) = \mathbb{E}\varphi(x + w(t))$, where $w(t)$ is a standard Wiener process, and for the integro-differential operator $\mathcal{A}$, that acts as $\mathcal{A}\varphi(x) = \int_{\mathbb{R}}(\varphi(x+y) - \varphi(x))\Lambda(dy)$, (or as $\mathcal{A}\varphi(x) = \int_{\mathbb{R}}(\varphi(x+y) - \varphi(x) - y\varphi'(x)\mathbf{1}_{[-1,1]}(y))\Lambda(dy))$ we have

$$e^{t\mathcal{A}}\varphi(x) = \mathbb{E}\varphi(x + \xi(t)) \tag{1}$$

where $\xi(t)$ is a jump Lévy process. Note that the representation (1) can be directly generalized neither for higher order differential operators nor for integro-differential operators that include higher order derivatives of $\varphi$. The simplest explanation of this fact is based on the maximum principle. An analogy of the representation (1) was considered in a number of papers (see [F79, DF83]). In this representation instead of usual probability processes so-called pseudo-processes were used. Note that there is no probabilistic interpretation of the the pseudo-processes.

We construct a probabilistic representation of the operator exponent $e^{t\mathcal{A}}$, where $\mathcal{A}$ belongs to a class of pseudo-differential operators. First we describe this class of operators.

Let $g$ be a generalized function on $\mathbb{R}$, such that for every $\varepsilon > 0$ the restriction of $g$ on $\mathbb{R}_\varepsilon = \mathbb{R} \setminus (-\varepsilon, \varepsilon)$ is a finite signed measure that is $|g|(\mathbb{R}_\varepsilon) < \infty$, and at the point 0 the generalized function $g$ can have a singularity of a finite order. For every generalized function $g$ we construct an operator $\mathcal{A}_g$, by $\mathcal{A}_g f(x) = (g_y, f(x+y))$, where we denote the action of $g$ on $f$ with respect to the variable $y$ by $g_y$.

To construct a probabilistic representation of the operator exponent of $\mathcal{A}_g$ we consider two objects defined by the generalized function $g$. The first object is a probability space $(\Omega, \mathcal{F}, P_g)$. Next we consider a subset $\Omega^0$ of $\Omega$ (in all interesting cases $P_g(\Omega^0) = 0$) and on $\Omega^0$ instead of a probability measure we define a generalized function $L_g$ So our second object will be a triple $(\Omega^0, \mathcal{G}, L_g)$, where $\mathcal{G}$ is a set of test functions of $L_g$.

All random processes we define on the space $\Omega^0$, and in the classical representation (1) instead of the mathematical expectation we use the generalized function $L_g$ (for the same functional). Then we study the connection between the probability measure $P_g$ and the generalized function $L_g$.

[DF83] Yu.L.Daletskii, S.V.Fomin. *Measures and differential equations in functional spaces.* Moscow: Nauka, 1983.

[F79] T.Funaki. *Probabilistic construction of the solution of some higher order parabolic differential equations.* Proc. Japan Acad. A 55, 176-179, 1979.