A sharp estimate for accuracy of short asymptotic expansions in the CLT in a Hilbert space

Vladimir V. Ulyanov[1] and Friedrich Götze[2]

Let $X, X_1, X_2, \dots$ be independent identically distributed random elements with values in a real separable Hilbert space H. Let (x, y) for $x, y \in H$ denote the inner product in H and put $|x| = (x, x)^{1/2}$. We assume that $\mathbf{E}|X_1|^2 < \infty$ and denote by V a covariance operator of X_1

$$(Vx, y) = \mathbf{E}(X_1 - \mathbf{E}X_1,\, x)(X_1 - \mathbf{E}X_1,\, y).$$

Let $\sigma_1^2 \geq \sigma_2^2 \geq \dots$ be the eigenvalues of V and let $e_1, e_2, \dots$ be the corresponding eigenvectors which we assume to be orthonormal. We define

$$S_n = n^{-1/2} \sigma^{-1} \sum_{i=1}^{n} (X_i - \mathbf{E}X_i),$$

where $\sigma^2 = \mathbf{E}|X_1 - \mathbf{E}X_1|^2$. For any integer $k > 0$ we put

$$c_k(V) = \prod_{1}^{k} \sigma_i^{-1}. \tag{1}$$

Let Y be H-valued Gaussian $(0, V)$ random element independent of X. Put for any $a \in H$

$$F(x) = P\{|S_n - a|^2 \leq x\}, \quad F_0(x) = P\{|Y - a|^2 \leq x\},$$

Define $F_1(x)$ as the unique function satisfying $F_1(-\infty) = 0$ with Fourier-Stieltjes transform equal to

$$\hat{F}_1(t) = -\frac{2t^2}{3\sqrt{n}} \mathbf{E} \exp\{it|Y - a|^2\} \left(3(X, Y - a)|X|^2 + 2it(X, Y - a)^3\right).$$

Introduce the error

$$\Delta_n(a) = \sup_x |F(x) - F_0(x) - F_1(x)|.$$

Theorem. *There exists an absolute constants c such that for any $a \in H$*

$$\begin{aligned} \Delta_n(a) \quad \leq \quad & \frac{c}{n} \cdot c_{12}(V) \cdot \big(\mathbf{E}|X_1|^4 + \mathbf{E}(X_1, a)^4\big) \\ & \times \big(1 + (Va, a)\big), \end{aligned} \tag{2}$$

where $c_{12}(V)$ is defined in (1).

For earlier versions of this result with 12 eigenvalues and a detailed discussion of the connection of the rate of convergence problem in CLT with classical lattice point problem in analytic number theory see Esseen (1945), the ICM-1998 Proceedings paper by Götze (1998), Götze and Ulyanov (2000) and Bogatyrev, Götze and Ulyanov (2006).

[1]Faculty of Computational Mathematics and Cybernetics, Moscow State University, Moscow 119991, Russia. Email: vladim53@yandex.ru

[2]Faculty of Mathematics, Bielefeld University, D-33501 Bielefeld 1, Germany. Email: goetze@math.uni-bielefeld.de

It follows from Lemma 2.6 in Götze and Ulyanov (2000) that for any given eigenvalues $\sigma_1^2, \dots, \sigma_{12}^2 > 0$ of a covariance operator V there exist $a \in H$, $|a| > 1$, and a sequence $X_1, X_2, \dots$ of i.i.d. random elements in H with zero mean and covariance operator V such that

$$\liminf_{n\to\infty} n\, \Delta_n(a) \geq c\, c_{12}(V)\, (1 + |a|^6)\, \mathbf{E}|X_1|^4.$$

Hence, in infinite dimensional H inequality (2) gives a bound which is the best possible one with respect to dependence on n, on the moments of X_1 and on the form of dependence in terms of V.

Similar problem in finite dimensional space H, where situation is essentially different, is considered in Götze and Zaitsev (2010).

[1] Bogatyrev, S.A., Götze, F., Ulyanov, V.V.: Non-uniform bounds for short asymptotic expansions in the CLT for balls in a Hilbert space. J. Multivariate Anal. **97**, 9, 2041–2056 (2006)

[2] Esseen, C.G.: Fourier analysis of distribution functions. Acta Math. **77**, 1–125 (1945)

[3] Götze, F.: Lattice point problems and the central limit theorem in Euclidean spaces. Doc. Math. J.DMV, Extra Vol. ICM, III, 245–255 (1998)

[4] Götze, F., Ulyanov, V.V.: Uniform approximations in the CLT for balls in Euclidian spaces, Preprint 00-034 SFB 343, Univ.Bielefeld (2000).

[5] Götze, F., Zaitsev, A. Yu.: Explicit rates of approximation in the CLT for quadratic forms. Preprint 10086. SFB 701, Univ.Bielefeld (2010).