On large deviations in stochastic averaging

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We present the large deviation principle for stochastic differential equations with averaging in the case when all coefficients of the fast component depend on the slow one, including diffusion. An SDE system is considered,

$$\begin{aligned} dX_t &= f(X_t, Y_t)dt, \quad X_0 = x_0, \\ dY_t &= \varepsilon^{-2} B(X_t, Y_t)dt + \varepsilon^{-1} C(X_t, Y_t)dW_t, \quad Y_0 = y_0. \end{aligned} \tag{1}$$

Here $X_t \in E^d$, $Y_t \in M$, M is a compact manifold of dimension ℓ (e.g. torus T^ℓ), f is a function with values in d–dimensional Euclidean space E^d, B is a function with values in TM, C is a function with values in $(TM)^\ell$ (i.e., in local coordinates an $\ell \times \ell$ matrix), W_t is an ℓ–dimensional Wiener process on some probability space (Ω, F, P), $\varepsilon > 0$ is a small parameter. Concerning SDE's on manifolds we refer to Watanabe and Ikeda (1989).

The large deviation principle (LDP) for such systems in the case $C(Y_t)$ was considered in papers by Freidlin (1976), Freidlin (1978), Freidlin and Wentzell (1984) for a compact state space and by Veretennikov (1994) for a non-compact one. The case with a "full dependence", that is, with $C(X_t, Y_t)$, was considered in Veretennikov (1999) and Veretennikov (2005). We present here the last corrected version. We do not mention some recent papers on more general systems with small additive diffusions, which so far only concerned the case $C(Y_t)$.

The LDP for systems like (1) is important in averaging and homogenization, in the KPP equation theory, for stochastic approximation algorithms with averaging, etc.

The family of processes X^ε satisfies a large deviation principle in the space $C([0,T]; R^d)$ with a normalizing coefficient ε^{-2} and a rate function $S(\varphi)$ if three conditions are satisfied:

$$\begin{aligned} \limsup_{\delta \to 0} \limsup_{\varepsilon \to 0} \varepsilon^2 \log P_x(X^\varepsilon \in F) &\le - \inf_F S(\varphi), \quad \forall F \text{ closed}, \\ \liminf_{\delta \to 0} \liminf_{\varepsilon \to 0} \varepsilon^2 \log P_x(X^\varepsilon \in G) &\ge - \inf_G S(\varphi), \quad \forall G \text{ open}, \end{aligned}$$

and S is a "good" rate function; that is, for any $s \ge 0$, the set

$$\Phi(s) := (\varphi \in C([0,T]; R^d) : \; S(\varphi) \le s, \; \varphi(0) = x)$$

is compact in $C([0,T]; R^d)$.

Let $\tilde W_t = \varepsilon^{-1} W_{t\varepsilon^2}$, $y_t = Y_{t\varepsilon^2}$, $x_t = X_{t\varepsilon^2}$ and let y_t^x denote a solution of SDE,

$$dy_t^x = B(x, y_t^x)dt + C(x, y_t^x)d\tilde W_t, \quad y_0^x = y_0.$$

The following assumptions are assumed.

(A_f) The function f is bounded and Lipschitz.

(A_C) The function CC^* is bounded, uniformly nondegenerate, C is Lipschitz.

(A_B) The function B is bounded and Lipschitz.

Theorem. *Let (A_f), (A_B), $A_C)$ be satisfied. Then the family $(X^\varepsilon_t = X_t,\ 0 \le t \le T)$ satisfies the LDP as $\varepsilon \to 0$ in the space $C([0,T]; R^d)$ with an action function*

$$S(\varphi) = \int_0^T L(\varphi_t, \dot\varphi_t)dt,$$

where

$$L(x,\alpha) = \sup_\beta(\alpha\beta - H(x,\beta)),$$

$$H(x,\beta) = \lim_{t\to\infty} t^{-1}\log E\exp\left(\int_0^t f(x, y^x_s)ds\right).$$

The limit H exists and is finite for any β, the functions H and L are convex in their last arguments β and α correspondingly, $L \ge 0$ and H is continuously differentiable in β.

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