Günter Last
Institut für Stochastik
Karlsruher Institut für Technologie

DFG FORSCHERGRUPPE
Geometry & Physics of
Spatial Random Systems

# How to find an extra excursion

Günter Last (Karlsruhe)

joint work with

Wenpin Tang (Berkeley) and Hermann Thorisson (Reykjavik)

presented at the VI-th conference

Modern Problems in Theoretical and Applied Probability

Sobolev Institute of Mathematics

Novosibirsk, August 22-28, 2016

# 1. Four problems on random shifts

## 1. Extra head problem

Consider a two-sided sequence of independent and fair coin tosses. Find a coin that landed heads so that the other coin tosses (centered around the picked one) are still independent and fair.

## 2. Marriage of Lebesgue and Poisson

Let $\eta$ be a stationary Poisson process in $\mathbb{R}^d$. Find a point $T$ of $\eta$ such that

$$\theta_T \eta \overset{d}{=} \eta + \delta_0.$$

## 3. Poisson matching

Let $\eta$ and $\xi$ be two independent stationary Poisson processes with equal intensity. Find a point $T$ of $\xi$ such that

$$\theta_T(\eta + \delta_0, \xi) \stackrel{d}{=} (\eta, \xi + \delta_0)$$

## 4. Unbiased embedding of Brownian excursions

Let $B = (B_t)_{t\in\mathbb{R}}$ be a two-sided standard Brownian motion. Let $A$ be a property of an excursion with positive and finite Itô measure. Find a random time $T$ with $B_T = 0$ such that the shifted process $B = (B_{T+t})_{t\in\mathbb{R}}$ splits into three independent pieces: a time reversed Brownian motion on $(-\infty, 0]$, an excursion distributed according to the conditional Itô law (given $A$) and a Brownian motion starting after this excursion.

# 2. Invariant transports of random measures

## Setting

$(\Omega, \mathcal{F}, \mathbb{P})$ is a $\sigma$-finite measure space.

## Definition

A random measure on $\mathbb{R}$ is a random element in the space of all locally finite measures on $\mathbb{R}$ equipped with the Kolmogorov product $\sigma$-field.

## Setting

We consider mappings $\theta_s : \Omega \to \Omega$, $s \in \mathbb{R}$, satisfying $\theta_0 = \mathrm{id}_\Omega$ and the flow property

$$\theta_s \circ \theta_t = \theta_{s+t}, \quad s, t \in \mathbb{R}.$$

The mapping $(\omega, s) \mapsto \theta_s\omega$ is supposed to be measurable. We assume that $\mathbb{P}$ is stationary, that is

$$\mathbb{P} \circ \theta_s = \mathbb{P}, \quad s \in \mathbb{R}.$$

## Definition

A random measure $\xi$ is invariant if the following holds for $\mathbb{P}$-a.e. $\omega \in \Omega$:

$$\xi(\theta_s\omega, B - s) = \xi(\omega, B), \quad s \in \mathbb{R}, B \in \mathcal{B}.$$

## Definition

Let $\xi$ be an invariant random measure on $\mathbb{R}$. The measure

$$\mathbb{P}_\xi(A) := \iint \mathbf{1}\{\theta_s\omega \in A, s \in B\}\, \xi(\omega, ds)\, \mathbb{P}(d\omega), \quad A \in \mathcal{F},$$

is called the Palm measure of $\xi$ (with respect to $\mathbb{P}$), where $B \in \mathcal{B}$ satisfies $\lambda_1(B) = 1$.

## Theorem (Refined Campbell theorem)

*Let $\xi$ be an invariant random measure on $\mathbb{R}$. Then*

$$\mathbb{E}_{\mathbb{P}} \int f(\theta_s, s)\, \xi(ds) = \mathbb{E}_{\mathbb{P}_\xi} \int f(\theta_0, s)\, ds$$

*for all measurable $f : \Omega \times \mathbb{R} \to [0, \infty)$.*

## Definition

An allocation is a measurable mapping $\tau : \Omega \times \mathbb{R} \to \mathbb{R}$ that is equivariant in the sense that

$$\tau(\theta_t \omega, s - t) = \tau(\omega, s) - t, \quad s, t \in \mathbb{R}, \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

## Theorem (L. and Thorisson '09)

*Let $\xi$ and $\eta$ be two invariant random measures with positive and finite intensities. Let $\tau$ be an allocation and define $T := \tau(\cdot, 0)$. Then*

$$\mathbb{P}_\xi(\theta_T \in \cdot) = \mathbb{P}_\eta$$

*iff $\tau$ is balancing $\xi$ and $\eta$, that is*

$$\int \mathbf{1}\{\tau(s) \in \cdot\} \xi(ds) = \eta \quad \mathbb{P}\text{-a.e.}$$

## Setting

$B = (B_t)_{t\in\mathbb{R}}$ is a two-sided standard Brownian motion with $B_0 = 0$, defined on its canonical probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$.

## Definition

For $t \in \mathbb{R}$ the shift $\theta_t \colon \Omega \to \Omega$ is given by

$$(\theta_t\omega)_s := \omega_{t+s}, \quad s \in \mathbb{R}.$$

For $x \in \mathbb{R}$ let

$$\mathbb{P}_x := \mathbb{P}_0(B + x \in \cdot), \quad x \in \mathbb{R},$$

where $B$ is the identity on $\Omega$.

## Definition

The local time measure $\ell^x$ at $x \in \mathbb{R}$ can be defined by

$$\ell^x(C) := \lim_{h \to 0} \frac{1}{h} \int \mathbf{1}\{s \in C, x \le B_s \le x + h\} ds.$$

Hence

$$\int f(B_s, s) ds = \iint f(x, s) \ell^x(ds) dx \quad \mathbb{P}_0\text{-a.s.}$$

for all measurable $f : \mathbb{R}^2 \to [0, \infty)$.

## Remark

It is a possible to choose a perfect version of local times, that is a (measurable) kernel satisfying for all $x \in \mathbb{R}$ and $\mathbb{P}_x$-a.e. that $\ell^x$ is diffuse and

$$\ell^x(\theta_t \omega, C - t) = \ell^x(\omega, C), \quad C \in \mathcal{B}, t \in \mathbb{R}, \mathbb{P}_x\text{-a.e. } \omega \in \Omega,$$

$$\ell^x(\omega, \cdot) = \ell^0(\theta_x \omega, \cdot), \quad \omega \in \Omega, x \in \mathbb{R},$$

$$\operatorname{supp} \ell^x(\omega) = \{t \in \mathbb{R} : B_t = x\}, \quad \omega \in \Omega, x \in \mathbb{R}.$$

## Definition

Let

$$\mathbb{P} := \int \mathbb{P}_x dx$$

be the distribution of a Brownian motion with a „uniformly distributed" starting value.

## Remark

Stationary increments of $B$ imply that $\mathbb{P}$ is stationary, that is

$$\mathbb{P} = \mathbb{P} \circ \theta_s, \quad s \in \mathbb{R}.$$

## Theorem (Geman and and Horowitz '73)

*The Palm (probability) measure of the local time $\ell^x$ is $\mathbb{P}_x$.*

## Definition

Let $\nu$ be a probability measure on $\mathbb{R}$. Define

$$\mathbb{P}_\nu := \int \mathbb{P}_x \nu(dx), \qquad \ell^\nu := \int \ell^x \nu(dx).$$

## Corollary

*$\mathbb{P}_\nu$ is the Palm probability measure of $\ell^\nu$.*

## Theorem (L., Mörters and Thorisson ’14)

*Let $\nu$ be a probability measure on $\mathbb{R}$ with $\nu\{0\} = 0$ and let*

$$T := \inf\{t > 0 \colon \ell^0[0, t] = \ell^\nu[0, t]\}.$$

*Then the process $(B_{T+t})_{t\in\mathbb{R}}$ has distribution $\mathbb{P}_\nu$. (In particular $B_T$ has distribution $\nu$.)*

## Remark

The above stopping time above was introduced in Bertoin and Le Jan (1992) as a solution of the Skorokhod embedding problem.

# 5. Embedding excursions

**Definition**

Let
$$R_t := \inf\{s > t : B_s = 0\} - t$$
denote the return time to 0 and consider the random measure
$$N := \sum_{s: R_{s-} \neq R_s} \delta_s.$$
For $N\{s\} > 0$ let the excursion $\epsilon_s$ starting at $s$ given by
$$\epsilon_s(t) := \begin{cases} B_{s+t}, & \text{if } 0 \leq t \leq R_s, \\ 0, & \text{if } t > R_s. \end{cases}$$

## Definition

The measure $\nu^I$ on the space $E$ of all excursions (space of all continuous functions starting and ending in 0 with a positive finite lifetime) is implicitly defined by

$$\mathbb{E} \int \mathbf{1}\{(s, \epsilon_s) \in \cdot\} \, N(ds) = \iint \mathbf{1}\{(s, e) \in \cdot\} \, ds \, \nu^I(de),$$

is Itô's excursion law (suitably normalized); see Pitman '87.

## Problem 4

Let $A \subset E$ be a measurable set such that $0 < \nu^l(A) < \infty$. Find a random time $T$ such that $\mathbb{P}_0(|T| < \infty) = 1$ and

$$\mathbb{P}_0(\theta_T B \in \cdot) = \mathbb{P}^- \odot \nu^l(\cdot|A) \odot \mathbb{P}^+,$$

where $\mathbb{P}^-$ (resp. $\mathbb{P}^+$) is the distribution of $(B_t)_{t \le 0}$ (resp. $(B_t)_{t \ge 0}$) and $\odot$ stands for independent concatenation.

## Theorem

*Let*

$$S_A := \inf\{t > 0 : t \in N, \epsilon_t \in A\}.$$

*Then* $\mathbb{P}_0(\epsilon_{S_A} \in \cdot) = \nu(\cdot \mid A)$*. However,* $(\theta_{S_A} B)^-$ *is not a time reversed Brownian motion.*

Proof of the second fact:

- By excursion theory $\ell^0[0, S_A]$ has an exponential distribution with rate $\nu(A)$.
- The same applies to $\ell^0[S'_A, 0]$, where

$$S'_A := \sup\{t < 0 : t \in N, \epsilon_t \in A\}.$$

- Hence $S'_A(\theta_{S_A} B)$ has a Gamma distribution with shape parameter 2.

## Theorem (L., Tang and Thorisson '16)

*Let $A \subset E$ be a measurable set such that $0 < \nu^l(A) < \infty$ and define*

$$N_A(C) := \frac{1}{\nu^l(A)} \int \mathbf{1}\{s \in C, \varepsilon_s \in A\}\, N(ds), \quad C \in \mathcal{B}(\mathbb{R}).$$

*Then*

$$T := \inf\{t > 0 \colon \ell^0[0,t] \le N_A[0,t]\}$$

*solves Problem 4.*

# 6. Ingredients of the proof

**Theorem (L., Mörters and Thorisson '14; L., Tang and Thorisson '16)**

*Let $\xi$ and $\eta$ be jointly stationary orthogonal random measures on $\mathbb{R}$ with finite and equal intensities. Assume that $\xi$ is diffuse. Then the mapping $\tau\colon \Omega \times \mathbb{R} \to \mathbb{R}$, defined by*

$$\tau(s) := \inf\{t > s : \xi[s, t] \leq \eta[s, t]\}, \quad s \in \mathbb{R},$$

*is an allocation balancing $\xi$ and $\eta$.*

**Theorem (Excursion Theory)**

*Let*

$$\tau_s := \inf\{t \geq 0 : \ell_t^0 > s\}, \quad s \geq 0,$$

*be the right-continuous generalized inverse of $\ell^0$. Then*

$$\Phi := \sum_{s>0:\tau_{s-}<\tau_s} \delta_{(s,\epsilon_{\tau_{s-}})}$$

*is a Poisson process on $(0,\infty) \times E$ under $\mathbb{P}_0$ with intensity measure $\lambda_+ \otimes \nu^l$.*

**Theorem (Pitman '87, L., Tang and Thorisson '16)**

*Let $A \subset E$ be such that $0 < \nu^l(A) < \infty$. The Palm measure of $N_A$ is given by $\mathbb{P}^- \odot \nu^l(\cdot|A) \odot \mathbb{P}^+$.*

# 6. References


- J. Bertoin and Y. Le Jan (1992). *Ann. Probab.* **20**, 538–548.
- A.E. Holroyd and Y. Peres (2005). Extra heads and invariant allocations. *Ann. Probab.* **33**, 31–52.
- G. Last, P. Mörters and H. Thorisson (2014). Unbiased shifts of Brownian motion. *Ann. Probab.* **42**, 431–463.
- G. Last, W. Tang and H. Thorisson (2016). Transporting random measures on the line and embedding excursions into Brownian motion. arXiv:1608.02016

- Last, G. and Penrose, M. (2016). *Lectures on the Poisson Process.* Cambridge University Press, to appear. `http://www.math.kit.edu/stoch/~last/seite/lehrbuch_poissonp/de`
- G. Last and H. Thorisson (2009). Invariant transports of stationary random measures and mass-stationarity. *Ann. Probab.* **37**, 790–813.
- T.M. Liggett (2002). Tagged particle distributions or how to choose a head at random. In *In and Out of Equlibrium* (V. Sidoravicious, ed.) 133–162, Birkhäuser, Boston.
- B. Mandelbrot (1982). *The Fractal Geometry of Nature.* Freeman and Co., San Francisco.
- J. Pitman (1987). Stationary excursions. *Séminaire de Probabilités, XXI*, *Lecture Notes in Math.*, **1247**, 289-302, Springer, Berlin.