The contact process on evolving scale-free networks

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joint work with

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Setup of the talk

(1) The contact process
(2) Scale-free networks
(3) The contact process on static scale-free networks
(4) The contact process on evolving scale-free networks
(5) Ideas, insights and method of proof

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After a random finite extinction time $T_{\rm ext}$ all vertices become healthy and remain so forever.

We start the process with all vertices infected and ask
How large is the extinction time?

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Quick extinction: The expected extinction time is at most polynomial in the number N of vertices in the network.

Slow extinction: With high probability the extinction time is at least exponential in the number N of vertices in the network.

Figure : Schematic energy landscape for quick and slow extinction. Slow extinction is due to metastability.

Scale-free networks

A feature of many networks is that they are scale-free, which means that for very large N and large k,

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Easiest model: The vertex set is $\{1, \ldots, N\}$ with small indices indicating large strength. Every pair of vertices connects independently and the probability of connecting the ith and jth indexed vertex in the network of size N is

$$p_{i,j} = \frac{\beta N^{2\gamma-1}}{i^\gamma j^\gamma},$$

where $\beta > 0$ and $\gamma \in (0,1)$ are the parameters of the model.

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The expected degree of the ith ranked vertex is $\sim \mathrm{const.}\big(\frac{N}{i}\big)^\gamma$ and therefore the power law exponent τ is given by $\tau = 1 + \frac{1}{\gamma}$.

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How does the behaviour change when the network is evolving?

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- Note that $\mathscr{G}_t \stackrel{d}{=} \mathscr{G}_0$ for all $t > 0$.

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Theorem: Jacob, M (2015)

Consider the contact process on the evolving network $(\mathscr{G}_t)_{t\geq 0}$, where at time $t=0$ every vertex is infected.

(a) If $\tau < 4$ (or equivalently $\gamma > 1/3$), then whatever the other parameters of the network, there exists $c > 0$ such that, uniformly in $N > 0$,

$$\mathbb{P}(T_{\mathrm{ext}} \leq e^{cN}) \leq e^{-cN}.$$

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Surprising observation: In the evolving network a quick extinction regime is possible, but only if $\tau > 4$, not if $\tau > 3$ as predicted by the mean field calculation.

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Observation: Updating can help the infection process to get out of metastable states. It therefore speeds up extinction.

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Observation: Updating helps the infection process to infect more vertices.
It therefore slows down extinction.

Method of proof: Existence of a quick extinction phase

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It is possible to couple the original process to the mean-field model in such a way that, at every time $t > 0$, every vertex which is *infected* in the original model, is either *ready* or *infected* in the mean-field model.

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Hence the extinction time in the mean-field model is a stochastic upper bound to the original extinction time.

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Extinction time in the mean-field model

If $\gamma < \frac{1}{3}$ and λ is small enough, the process

$$M(t) := \sum_{i=1}^{N} \mathbf{1}\{i \textit{ ready} \text{ at time } t\}\, s_1(i) + \sum_{i=1}^{N} \mathbf{1}\{i \textit{ infected} \text{ at time } t\}\, s_2(i)$$

with

$$s_1(i) = \left(\frac{N}{i}\right)^{2\gamma} \qquad\qquad s_2(i) = s_1(i) + \left(\frac{N}{i}\right)^{\gamma},$$

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Hence $(Z(t))_{0\le t<T_{\mathrm{ext}}}$ is a positive supermartingale, and we deduce

$$\mathbb{E}\, T_{\mathrm{ext}} \sim c^{-1} N^{\gamma} \mathbb{E}[Z(T_{\mathrm{ext}}^{-})] \le c^{-1} N^{\gamma} \mathbb{E} Z(0) = \mathcal{O}(N^{\gamma} \log N).$$

Thank you very much for your attention!