An $M/M/\infty$-type model for synchronization in the Bitcoin network

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Outline

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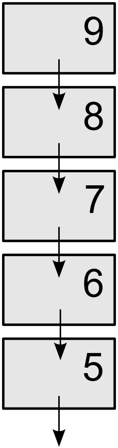
Bitcoin network in a nutshell

- Bitcoin network — payment system
- bitcoins — virtual currency
- global transaction history is stored in the blockchain
- every node maintains a blockchain version (transparency)

Blockchain

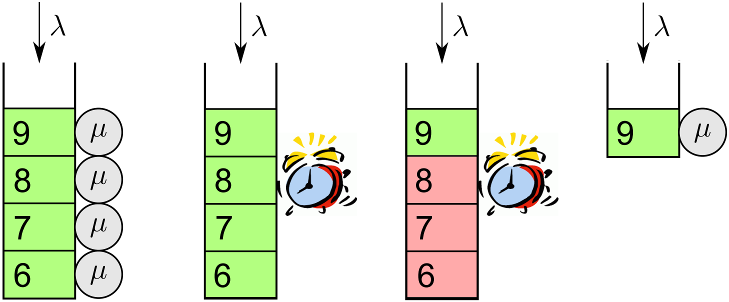


- next block contains a link to the previous one
- miners arrange transactions into blocks and broadcast blocks over the network

Modeling

- Part 1 generates blocks — rate λ
- Part 2 receives them delayed — rate μ per block
- $\#$ blocks Part 1 has generated
  but Part 2 has not received — $Q(\cdot)$

$M/M/\infty$ with FIFO batch departures

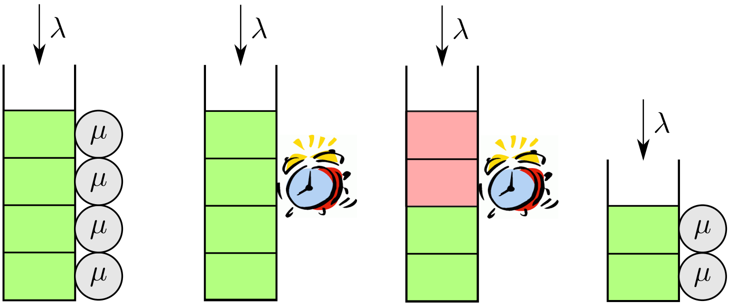


Outline

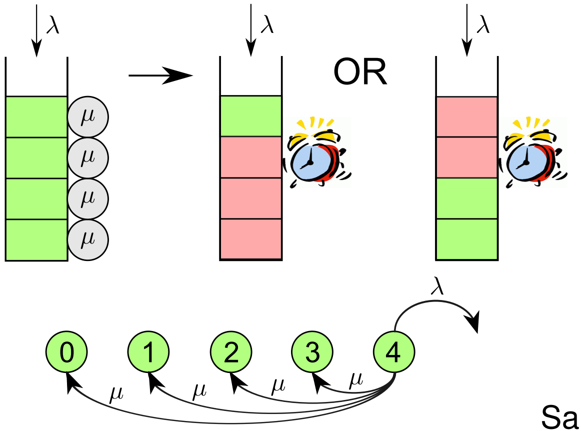
Busy period

Conventional $M/M/\infty$	Bitcoin $M/M/\infty$
$\mathbb{E}B = \dfrac{e^{\rho} - 1}{\lambda}$	$\mathbb{E}B = \dfrac{1}{\mu}$
$\mathbb{E}e^{-sB} = 1 + \dfrac{s}{\lambda} - \dfrac{e^{\rho}}{\lambda \int_0^{\infty} e^{-st+\rho e^{-\mu t}} dt}$	$B \sim \text{Exp}(\mu)$
	No $B \uparrow$ as $\lambda \uparrow$???

$M/M/\infty$ with LIFO batch departures

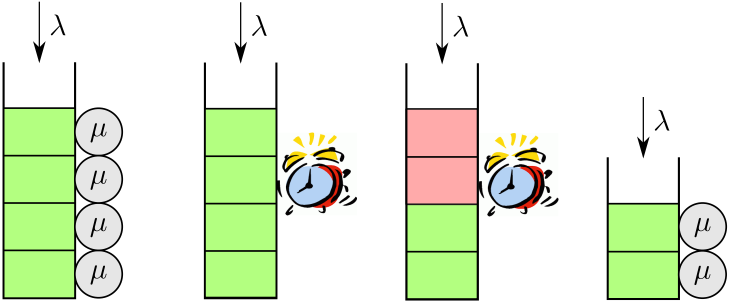


Equivalence



Same Busy Period!

Busy period under LIFO-batch departures



The busy period lasts while the initial customer is there!

Outline

Stationary distribution

Conventional $M/M/\infty$	Bitcoin $M/M/\infty$
$Q_\infty \sim \mathrm{Poi}(\rho)$	$\mathbb{P}\{Q_\infty \geq k\} = \frac{\rho}{\rho+1} \cdots \frac{\rho}{\rho+k}$
[1] $(m)_k = \rho^k$	[2] $(m)_{k+2} = (k+2)(\rho(m)_k - (m)_{k+1})$
	$\mathbb{E}Q_\infty \sim \sqrt{\frac{\pi}{2}}\sqrt{\rho}$ as $\rho \to \infty$

[1] $(m)_k = \mathbb{E}Q_\infty(Q_\infty - 1)\ldots(Q_\infty - k + 1)$
[2] via balance equations and generating function

Stationary distribution

As $\rho \to \infty$,

$$\frac{\mathbb{E}Q_\infty}{\sqrt{\rho}} \to \sqrt{\frac{\pi}{2}}$$

Moreover,

$$\frac{Q_\infty}{\sqrt{\rho}} \Rightarrow \xi,$$

where

$$\mathbb{P}\{\xi \geq x\} = \exp\left(-\frac{x^2}{2}\right), \quad x > 0$$

Outline

Transient behavior

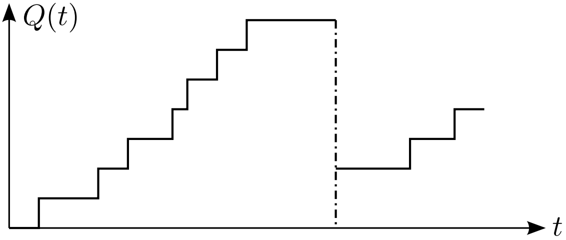
Solution to the Kolmogorov equations is quite implicit...

$$\mathbb{P}\{Q(t) = n\} = \mathbb{P}\{Q_\infty = n\} + \sum_{k=1}^{n+1} C_{n,k} e^{-(\lambda + k\mu)t}$$

where $C_{n,k}$ satisfy certain recursive relations

Scaling limit

- Service slows down: $\lambda^N = \lambda$, $\mu^N = \dfrac{\mu}{N}$, $\rho^N = \rho N$
- Hint for space scaling: $\mathbb{E}Q_\infty \sim \sqrt{\dfrac{\pi\rho}{2}}\sqrt{N}$
- Hint for time scaling:



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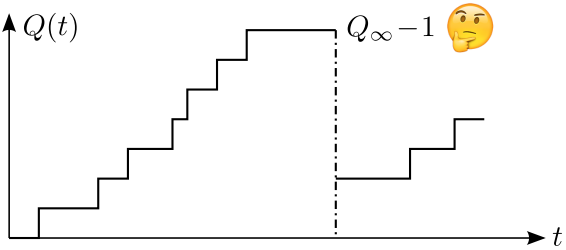

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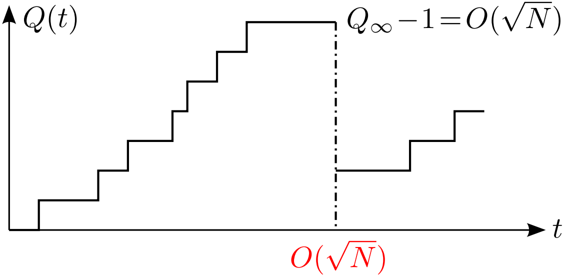

Scaling limit

As $N \to \infty$,

$$\left\{ \frac{Q(\sqrt{N}t)}{\sqrt{N}}, t \geq 0 \right\} \Rightarrow \text{LLN / fluid limit } \{\overline{Q}(t), t \geq 0\}$$

cf. FLLN: $X_1, X_2, \ldots$ — i.i.d.,

$$\left\{ \frac{X_1 + \ldots + X_{\lfloor Nt \rfloor}}{N}, t \geq 0 \right\} \Rightarrow \{\mathbb{E}X_1 t, t \geq 0\}$$

LLN limit

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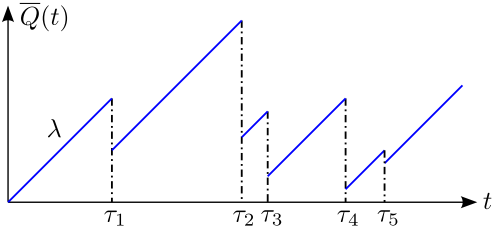
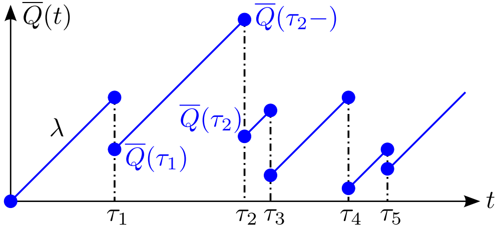


Random LLN limit!



- $\{\overline{Q}(\tau_{k-1}), \overline{Q}(\tau_k-), k \geq 1\}$ — MC
- $\overline{Q}(\tau_2) \sim \text{Unif}(0, \overline{Q}(\tau_2-))$
- $\mathbb{P}\{Q(\tau_2-) \geq y | \overline{Q}(\tau_1) = x\} = \exp\left(-\frac{y^2 - x^2}{2\rho}\right), y > x$
- $\tau_2 - \tau_1 = \dfrac{\overline{Q}(\tau_2-) - \overline{Q}(\tau_1)}{\lambda}$

Other examples of random LLN limits:

- Kovalevski, Topchii, Foss. On the stability of a queueing system with uncountable branching fluid limits. Problems of Information Transmission, 2005.
- Remerova, Foss, Zwart. Random fluid limit of an overloaded polling model. AAP, 2014.

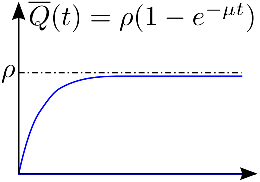
The reason: an embedded supercritical branching process.

Outline

Interchange of limits for the conventional $M/M/\infty$

Service slows down: $\lambda^N = \lambda$, $\mu^N = \dfrac{\mu}{N}$, $\rho^N = \rho N$

$$\begin{tikzcd}
\dfrac{Q(Nt)}{N} \arrow[r, "N\to\infty"] \arrow[d, "t\to\infty"'] & \overline{Q}(t) \arrow[d] \\
\dfrac{\mathrm{Poi}(\rho N)}{N} \arrow[r] & \rho
\end{tikzcd}$$

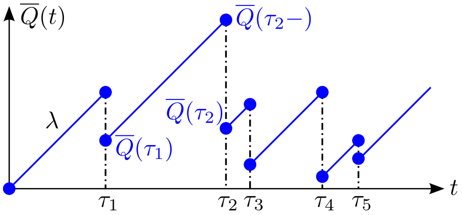


- Poi(ρN) is invariant and limit d-n for $M/M/\infty$
- ρ is the invariant and limit point for the fluid limit

$$\overline{Q}'(t) = \lambda - \mu\overline{Q}(t)$$

Bitcoin $M/M/\infty$

- $$\frac{Q(\sqrt{N}\cdot)}{\sqrt{N}} \Rightarrow \overline{Q}(\cdot)$$

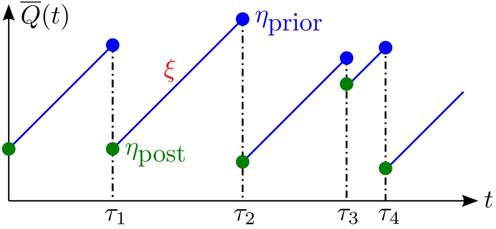


- $$\frac{Q_\infty}{\sqrt{N}} \Rightarrow \xi,$$

$$\mathbb{P}\{\xi \geq x\} = \exp\left(-\frac{x^2}{2\rho}\right), \quad x > 0$$

Where does ξ show in the fluid limit $\overline{Q}(\cdot)$?

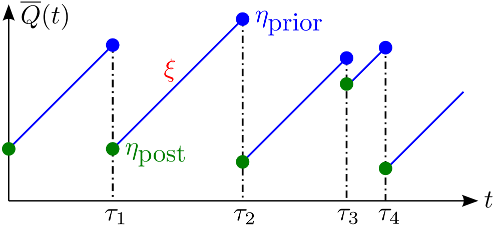
Bitcoin $M/M/\infty$



$$\mathbb{E}\xi = \frac{\mathbb{E}\eta_{\text{post}} + \mathbb{E}\eta_{\text{prior}}}{2}$$

- $(\eta_{\text{post}}, \eta_{\text{prior}})$ — stationary versions of $(\overline{Q}(\tau_k - 1), \overline{Q}(\tau_k -))$
- $f_{\eta_{\text{post}}}(x) = \sqrt{\frac{2}{\pi\rho}} \exp\left(-\frac{x^2}{2\rho}\right)$, $x > 0$
- $\mathbb{P}\{\eta_{\text{prior}} \geq y | \eta_{\text{post}} = x\} = \exp\left(-\frac{y^2 - x^2}{2\rho}\right)$, $y > x$
- ξ — “overall” stationary

Bitcoin $M/M/\infty$



$$\mathbb{E}\xi = \frac{\mathbb{E}\eta_{\text{post}} + \mathbb{E}\eta_{\text{prior}}}{2}$$

WRONG!!!

- $(\eta_{\text{post}}, \eta_{\text{prior}})$ — stationary versions of $(\overline{Q}(\tau_k - 1), \overline{Q}(\tau_k -))$
- $f_{\eta_{\text{post}}}(x) = \sqrt{\frac{2}{\pi\rho}} \exp\left(-\frac{x^2}{2\rho}\right)$, $x > 0$
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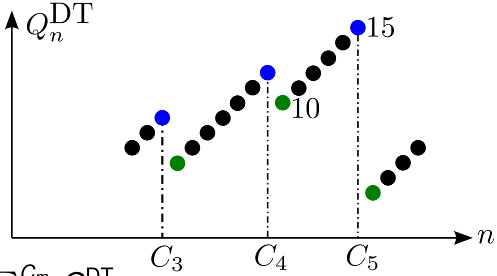
What is really going on

Prelimit system in discrete time:



$$\mathbb{E}Q_\infty^{\mathrm{DT}} \sim \frac{\sum_{i=1}^{n} Q_i^{\mathrm{DT}}}{n} \sim \frac{\sum_{i=1}^{C_m} Q_i^{\mathrm{DT}}}{C_m}$$

$$= \frac{\ldots + (10+15)(15-10+1)/2 + \ldots}{\ldots + (15-10+1) + \ldots}$$

$$\sim \frac{\mathbb{E}(Q_{\infty,\mathrm{prior}} + Q_{\infty,\mathrm{post}})(Q_{\infty,\mathrm{prior}} - Q_{\infty,\mathrm{post}} + 1)/2}{\mathbb{E}(Q_{\infty,\mathrm{prior}} - Q_{\infty,\mathrm{post}} + 1)}$$

What is really going on

In the pre-limit system:

$$\mathbb{E}Q_\infty \sim \mathbb{E}Q_\infty^{\mathrm{DT}} \sim \frac{\mathbb{E}(Q_{\infty,\mathrm{prior}} + Q_{\infty,\mathrm{post}})(Q_{\infty,\mathrm{prior}} - Q_{\infty,\mathrm{post}} + 1)/2}{\mathbb{E}(Q_{\infty,\mathrm{prior}} - Q_{\infty,\mathrm{post}} + 1)}$$

Scale by $\sqrt{N}$, in the LLN limit:

$$\mathbb{E}\xi = \frac{\mathbb{E}(\eta_{\mathrm{prior}} + \eta_{\mathrm{post}})(\eta_{\mathrm{prior}} - \eta_{\mathrm{post}})/2}{\mathbb{E}(\eta_{\mathrm{prior}} - \eta_{\mathrm{post}})}$$

- this part is in progress
- want to: relate ξ, η_{post}, η_{prior} distributionally
- want to: η_{post}, η_{prior} are LLN limits of $Q_{\infty,\text{post}}$, $Q_{\infty,\text{prior}}$ (simulations suggest so)

Thank you!