Dynamics of Large-Scale Spiking Neural Networks

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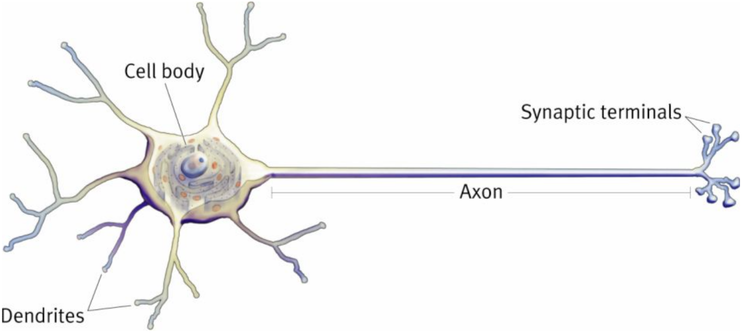
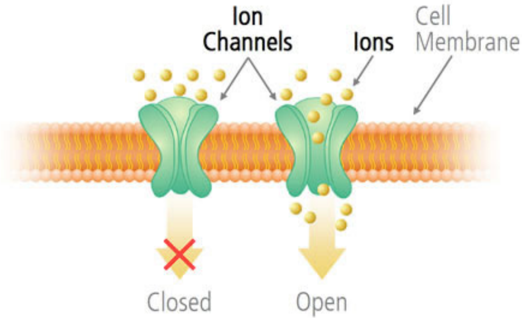
Introduction

Neuronal Cell



Ion Channels



Dynamics

Simple Classical Model

Evolution of membrane potential of a neuron :

- **increases with external inputs of other neurons**

Dynamics

Simple Classical Model

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- decays at some rate

Dynamics

Simple Classical Model

Evolution of membrane potential of a neuron :

- **increases with external inputs of other neurons**
- **decays at some rate**
- **if fixed threshold is hit :**
  - **input sent to neighboring neurons**
  - **returns to some fixed value**

Mathematical Models

Empirical model : Hodgkin-Huxley (1952)

$$\dot{V} = g_{\mathrm{Na}} m^3 h (V - V_{\mathrm{Na}}) + g_{\mathrm{K}} n^4 (V_{\mathrm{K}} - V) + g_{\mathrm{L}} (V_{\mathrm{L}} - V) + I_e$$

$$\begin{aligned}
\dot{m} &= \alpha_m(V)(1-m) - \beta_m(V)m \\
\dot{h} &= \alpha_h(V)(1-m) - \beta_h(V)m \\
\dot{n} &= \alpha_n(V)(1-m) - \beta_n(V)m
\end{aligned}$$

Mathematical Models

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$$\begin{aligned}\dot{m} &= \alpha_m(V)(1 - m) - \beta_m(V)m \\ \dot{h} &= \alpha_h(V)(1 - m) - \beta_h(V)m \\ \dot{n} &= \alpha_n(V)(1 - m) - \beta_n(V)m\end{aligned}$$

Nobel Prize (1963)

Other Mathematical Models

Empirical models

- **FitzHugh-Nagumo**
- **Wilson-Cowan**
- **Morris-Lecar**
- **. . .**

A classsical Model

Leaky integrate and fire model (LIFM)

Simple ODE for Potential $X_i(t)$ of neuron i

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if $X_i(t) < V_F$

$$\mathrm{d}X_i(t) = -X_i(t)\mathrm{d}t + \sum_{j \neq i} W_{ji}(t)\mathrm{d}D_j(t) + \mathrm{d}I_i^e(t),$$

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- $\mathrm{d}D_j(t)$ **spikes from node** j
- $W_{ji}(\cdot)$ **output from node** j **to** i
- I_i^e **external input to** i

A classsical Model

Leaky integrate and fire model (LIFM)

Simple ODE for Potential $X_i(t)$ of neuron i

if $X_i(t) < V_F$

$$\mathrm{d}X_i(t) = -X_i(t)\mathrm{d}t + \sum_{j \neq i} W_{ji}(t)\mathrm{d}D_j(t) + \mathrm{d}I_i^e(t),$$

if $X_i(t-) = V_F$ then $\mathrm{d}D_i(t) = 1,$

$X_i(t) = V_R$

- $\mathrm{d}D_j(t)$ spikes from node j
- $W_{ji}(\cdot)$ output from node j to i
- I_i^e external input to i

Stochastic Versions of LIFM

Where is the randomness?

- I_i^e external input to i is Gaussian

$$\mathrm{d}I_i^e(t) = \mathrm{d}B_i(t)$$

  a common assumption in literature

Stochastic Versions of LIFM

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- Instants of spikes

  Instead of deterministic threshold V_R

  Randomness in firing instants

  Chichilnisky (2001), Pillow et al. (2005)

  Non-Linear Poisson Model

A stochastic LIFM

Main Assumptions

- external input may be zero
- randomness of firing instants
  a neuron i in state x fires at rate $b(x)$

A stochastic LIFM

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  a neuron i in state x fires at rate $b(x)$
  $x \mapsto b(x) \nearrow$

Outline of the talk

- **Stability Properties of Finite Networks**

- **Mean-Field Limits**

- **Invariant Distributions**

Finite Networks

Stochastic Model

N **nodes**

Node in state x fires at rate $b(x)$

$\mathcal{N}_{b(y)}(\mathrm{d}t)$ **Poisson process with rate $b(y)$**

$$\mathrm{d}X_i(t) = -X_i(t)\mathrm{d}t + \sum_{j\neq i} W_{ji}(t)\mathcal{N}^j_{b(X_j(t))}(\mathrm{d}t) - X_i(t)\mathcal{N}^i_{b(X_i(t))}(\mathrm{d}t)$$

- (W_{ij}) **i.i.d. random variables**

Stability of Finite Networks

Theorem

- **If** $b(0) > 0$ **:** $(X_i(t), 1 \leq i \leq N)$

  Ergodic Markov process

  with non-trivial invariant distribution

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- If $b(0) = 0$ and
  $$\int_0^1 \frac{b(s)}{s}\,\mathrm{d}s < +\infty$$
  no spike occurs after a finite (random) time

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  δ_0 is the unique invariant distribution

On the condition $\mathbf{b(0) > 0}$

Spiking process :

sum of two ind. Poisson processes

$$\mathcal{N}_{b(y)}(\mathrm{d}t) \stackrel{\text{dist.}}{=} \mathcal{N}_{b(0)}(\mathrm{d}t) + \mathcal{N}_{b(y)-b(0)}(\mathrm{d}t)$$

On the condition $\mathbf{b(0) > 0}$

Spiking process :

sum of two ind. Poisson processes

$$\mathcal{N}_{b(y)}(\mathrm{d}t) \overset{\text{dist.}}{=} \overbrace{\mathcal{N}_{b(0)}(\mathrm{d}t)}^{\text{Ext. Input}} + \overbrace{\mathcal{N}_{b(y)-b(0)}(\mathrm{d}t)}^{\text{State dependent}}$$

Large Networks

Mean-Field Asymptotics

Large Networks

Technical Assumptions

- $x \mapsto b(x) \nearrow C^1$**-function**

$$b'(x) \leq \gamma b(x) + c$$

- $W_{ij} = V_{ij}/N$ **with** (V_{ij}) **i.i.d. with bounded support and**

$$3\mathbb{E}(V)\gamma < 1$$

Large Networks : Asymptotics

$$\begin{aligned}\mathrm{d}X_i^N(t) = -X_i^N(t)\mathrm{d}t - X_i^N(t)\mathcal{N}^i_{b(X_i^N(t))}(\mathrm{d}t) \\ + \sum_{j=1}^{N} W_{ji}(t)\mathcal{N}^j_{b(X_j^N(t))}(\mathrm{d}t)\end{aligned}$$

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Mean-Field Approximation

$$\frac{1}{N}\sum_{j=1}^{N} V_{ji}(t)\mathcal{N}^j_{b(X_j^N(t))}(\mathrm{d}t) \sim \mathbb{E}(V)\mathbb{E}(b(X_1^N(t)))\,\mathrm{d}t$$

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McKean-Vlasov Process

Large Networks : main steps

- **Analysis of McKean-Vlasov process**

- **Proof of the Mean-Field Convergence**

Literature : Mean-Field Results

Poissonnian Setting

- **An empirical model**

  Fournier, Löcherbach

  de Masi, Galves, Löcherbach, Presutti

$$\mathrm{d}X_i(t) = -\left(X_i(t) - \frac{1}{N}\sum_{j=1}^{N} X_j(t)\right)\mathrm{d}t$$

$$+\sum_{j\neq i} W_{ji}(t)\mathcal{N}^{j}_{b(X_j(t))}(\mathrm{d}t) - X_i(t)\mathcal{N}^{i}_{b(X_i(t))}(\mathrm{d}t)$$

  Model with entropy function

McKean-Vlasov process

Theorem

There exists a unique process $(Z(t))$ such that

$$dZ(t) = -Z(t)\mathrm{d}t + \mathbb{E}(V)\mathbb{E}(b(Z(t)))\mathrm{d}t \\ - Z(t-)\mathcal{N}_{b(Z(t-))}(\mathrm{d}t)$$

and $Z(0) = x > 0$

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and $Z(0) = x > 0$

Proof

Iteration scheme + stochastic calculus

Mean-Field Results

Empirical Distribution

$$\langle \Lambda_N(t), \phi \rangle = \frac{1}{N} \sum_{i=1}^{N} \phi(X_i^N(t)).$$

Mean-Field Results

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Fraction of nb of nodes with potential $\leq x$

$$\left\langle \Lambda_N(t), \mathbb{1}_{[0,x]} \right\rangle$$

Mean-Field Results

Empirical Distribution

$$\langle \Lambda_N(t), \phi \rangle = \frac{1}{N} \sum_{i=1}^{N} \phi(X_i^N(t)).$$

Mean-Field result :

If $(Z(t))$ is the McKean-Vlasov process

$$\lim_{N \to +\infty} \left(\langle \Lambda_N(t), \phi \rangle, t \geq 0 \right) = \left(\mathbb{E}(\phi(Z(t))), t \geq 0 \right)$$

Mean-Field Results

Empirical Distribution

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Mean-Field result :

If $(Z(t))$ is the McKean-Vlasov process

$$\lim_{N \to +\infty} \left(\langle \Lambda_N(t), \phi \rangle , t \geq 0 \right) = \left(\mathbb{E}(\phi(Z(t))), t \geq 0 \right)$$

or, for $i \geq 1$

$$\lim_{N \to +\infty} \left(X_i^N(t), t \geq 0 \right) = (Z(t), t \geq 0)$$

Mean-Field Results

Technical Difficulties

$$
\begin{aligned}
\mathrm{d}X_i^N(t) = -X_i^N(t)\mathrm{d}t - X_i^N(t)\mathcal{N}^i_{b(X_i^N(t))}(\mathrm{d}t) \\
+ \frac{1}{N}\sum_{j\neq i} V_{ji}(t)\mathcal{N}^j_{b(X_j^N(t))}(\mathrm{d}t)
\end{aligned}
$$

$b(\cdot)$ **NOT** **Lipschitz**

Important example $b(x)=x^2$

Mean-Field Results

Technical Difficulties

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$b(\cdot)$ **NOT Lipschitz**

Important example $b(x)=x^2$

$\Rightarrow$ **pb in interaction term**

Mean-Field without Lipschitz

Non-Lipschitz setting

- Scheutzow (1987)
- Malrieu (2003),
  Cattiaux, Guillin, Malrieu (2006)
- Bolley, Cañizo and Carillo (2011)
- ...

Proof of Mean-Field in our case

- Stochastic calculus with Poisson processes

- **Uniform bounds for moments of the scaled global firing rate**

$$\langle \Lambda_N(t), b^p \rangle = \frac{1}{N}\sum_{i=1}^{N} b(X_i(t)),$$

Proof of Mean-Field in our case

- Stochastic calculus with Poisson processes

- Uniform bounds for moments of the scaled global firing rate

$$\langle \Lambda_N(t), b^p \rangle = \frac{1}{N} \sum_{i=1}^{N} b^p(X_i(t)),$$

$\exists \delta > 3$ such that

$$\sup_{N \geq 1} \sup_{t \geq 0} \mathbb{E}\left(\langle \Lambda_N(t), b^\delta \rangle \right) < +\infty$$

Invariant States

of McKean Vlasov Processes

McKean-Vlasov process : Equilibrium

Stochastic Differential Equation

$$
\begin{aligned}
\mathrm{d}Z(t) = -Z(t)\mathrm{d}t + \mathbb{E}(V)\mathbb{E}(b(Z(t)))\mathrm{d}t \\
- Z(t)\mathcal{N}_{b(Z(t))}(\mathrm{d}t)
\end{aligned}
$$

McKean-Vlasov process : Equilibrium

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if at equilibrium

$$\mathbb{E}(b(Z(t))) = \mathbb{E}(b(Z(0))) \stackrel{\text{def.}}{=} \alpha, \quad \forall t \geq 0$$

McKean-Vlasov process : Equilibrium

Stochastic Differential Equation

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Starting from 0, evolves as

$$\dot{z} = \alpha\mathbb{E}(V) - z$$

until it jumps back to 0 at rate $b(z)$

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Remember that $\alpha = E(b(Z(0)))$

McKean-Vlasov process : Equilibrium

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until it jumps back to 0 at rate $b(z)$

Remember that $\alpha = E(b(Z(0)))$

$b(0) = 0 \Rightarrow \delta_0$ **is invariant**

McKean-Vlasov process : Equilibrium

$$\dot{z}_\alpha = \alpha \mathbb{E}(V) - z_\alpha \quad \textbf{with } z(0) = 0$$

McKean-Vlasov process : Equilibrium

Instant of first spike τ

$$\mathbb{P}(\tau \geq t) = \exp\left(-\int_0^t b(z_\alpha(u))\,\mathrm{d}u\right)$$

McKean-Vlasov process : Equilibrium

Instant of first spike τ

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Invariant distribution π_α **of** $(Z(t))$

$$\pi_\alpha(f) = \frac{1}{\mathbb{E}(\tau)}\mathbb{E}\left(\int_0^\tau f(z_\alpha(u))\,\mathrm{d}u\right)$$

McKean-Vlasov process : Equilibrium

Instant of first spike τ

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Invariant distribution π_α of $(Z(t))$

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$\alpha = E_{\pi_\alpha}(b(Z(0))) \Rightarrow$ **fixed point equation**

$$\alpha = \int b(u)\,\pi_\alpha(\mathrm{d}u)$$

McKean-Vlasov process : Equilibrium

Proposition If

$$C(\beta)=\int_0^1 \frac{1}{1-x}\exp\left(-\int_0^x \frac{b(\beta\mathbb{E}(V)y)}{1-y}\,\mathrm{d}y\right)\mathrm{d}x$$

if there exists $\beta_0>0$ satisfying **fixed point equation**

$$\beta_0 C(\beta_0)=1$$

McKean-Vlasov process : Equilibrium

Proposition If

$$C(\beta) = \int_0^1 \frac{1}{1-x} \exp\left(-\int_0^x \frac{b(\beta\mathbb{E}(V)y)}{1-y}\,\mathrm{d}y\right)\mathrm{d}x$$

if there exists $\beta_0 > 0$ satisfying fixed point equation

$$\beta_0 C(\beta_0) = 1$$

then distribution on $[0, \beta_0\mathbb{E}(V))$ with density

$$u \mapsto \frac{1}{C(\beta_0)(\beta_0\mathbb{E}(V)-u)} \exp\left(-\int_0^u \frac{b(s)}{\beta_0\mathbb{E}(V)-s}\,\mathrm{d}s\right)$$

is invariant

Examples

$$b(x) = \lambda x + \delta$$

If $\delta > 0$

- **unique non-trivial invariant distribution**

Examples

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Examples

$b(x) = \lambda x$

If $\lambda \mathbb{E}(V) < 1$

- δ_0 is the unique invariant distribution
- it is stable

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$b(x) = \lambda x$

If $\lambda \mathbb{E}(V) < 1$

- δ_0 is the unique invariant distribution
- it is stable

If $\lambda \mathbb{E}(V) > 1$

- $\exists$ non trivial invariant distribution
- inv. dist. δ_0 is not stable

The linear case : a summary

If $b(x) = \lambda x$ and $\overline{W} = \lambda \mathbb{E}(V) > 1$

- For $N \geq 1$

$$\lim_{t \to +\infty} (X_i^N(t), 1 \leq i \leq N) = 0$$

The linear case : a summary

If $b(x) = \lambda x$ and $\overline{W} = \lambda \mathbb{E}(V) > 1$

- For $N \geq 1$

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- But, in the limit,
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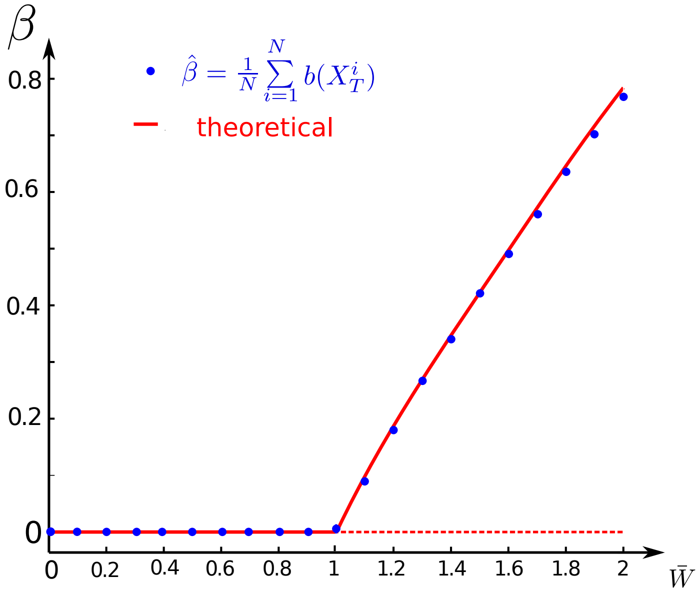
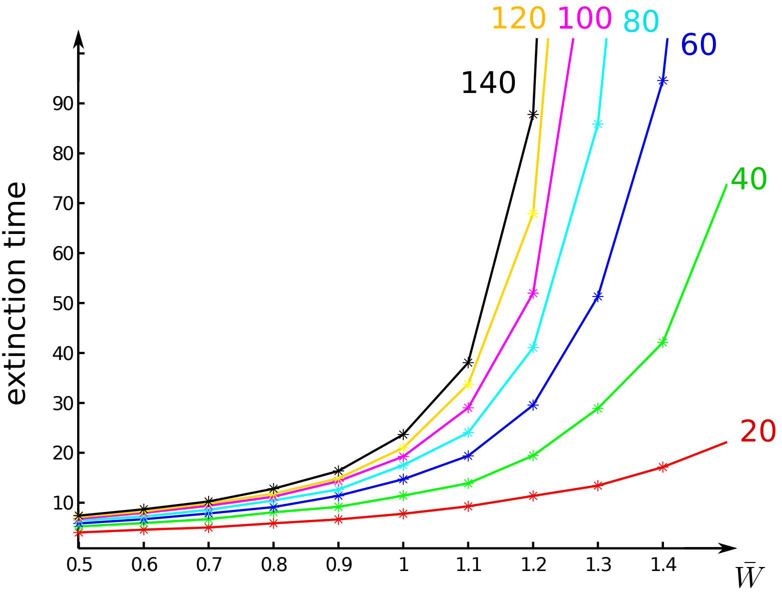
  - δ_0 is not stable

$\Rightarrow$ suggests a non-trivial quasi-stationary distribution before "death"

The linear case : mean firing rate



The linear case : Extinction Time



Examples

$b(x) = \lambda x^{\alpha}$ with $\alpha > 1$

Proposition

There exists some $\rho_c > 0$ such that

- If $\overline{W} = \lambda \mathbb{E}(V) < \rho_c$

  δ_0 is the unique invariant distribution

Examples

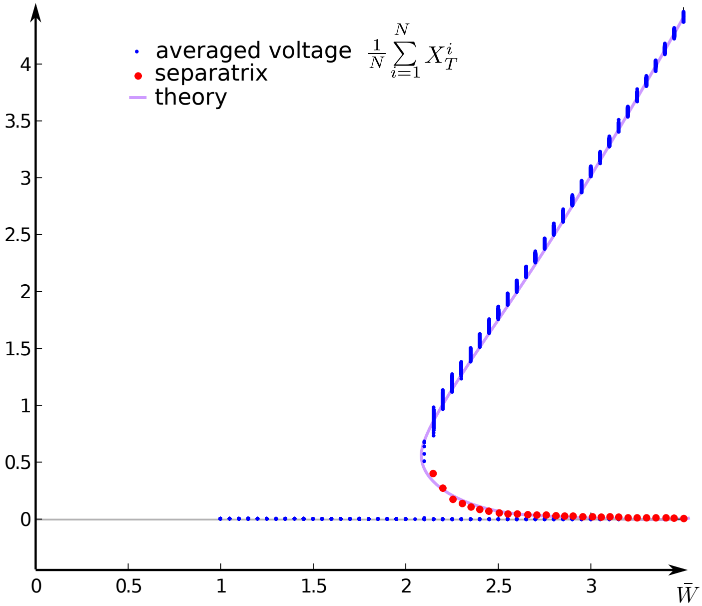
$b(x) = \lambda x^{\alpha}$ with $\alpha > 1$

Proposition

There exists some $\rho_c > 0$ such that

- If $\overline{W} = \lambda \mathbb{E}(V) < \rho_c$
  δ_0 is the unique invariant distribution

- If $\overline{W} > \rho_c$
  There exist at least TWO non-trivial invariant distributions


averaged voltage $\frac{1}{N}\sum_{i=1}^{N} X_T^i$
separatrix
theory
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
$\bar{W}$

Conclusion

Future work

- When $b(0) = 0,$ for a fixed N
  Phase transition for instant of last spike
  as $N \to +\infty$?

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Future work

- When $b(0) = 0$, for a fixed N
  Phase transition for instant of last spike
  as $N \to +\infty$?

- Stability properties of
  non-trivial invariant distributions

The End