Rumor spread and competition on scale-free random graphs

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Joint work with:

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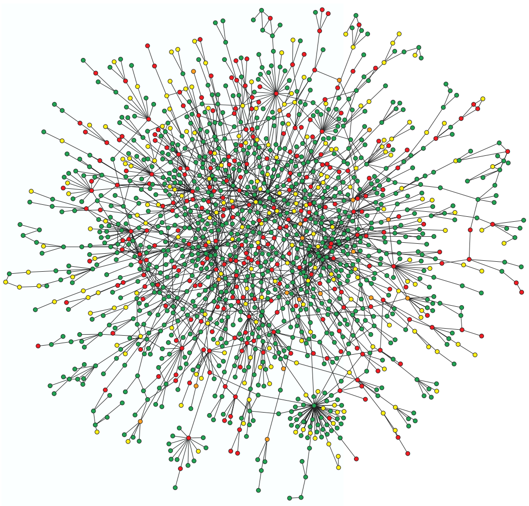
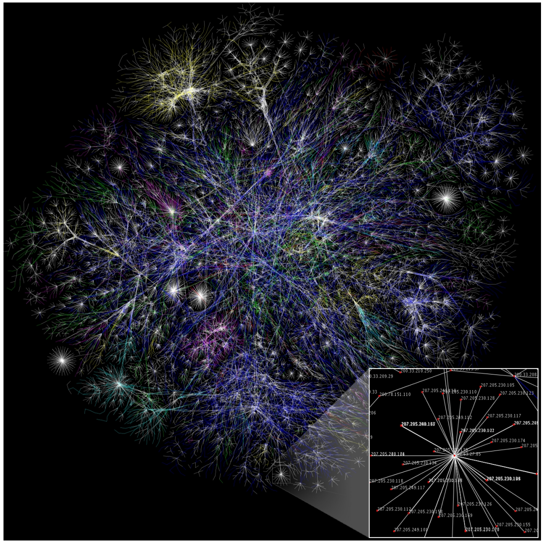
▷ Júlia Komjáthy (TU/e).

Builds on work with

▷ Shankar Bhamidi (North Carolina);
▷ Gerard Hooghiemstra (Delft);
▷ Dmitri Znamensky (Philips Research).

Complex networks

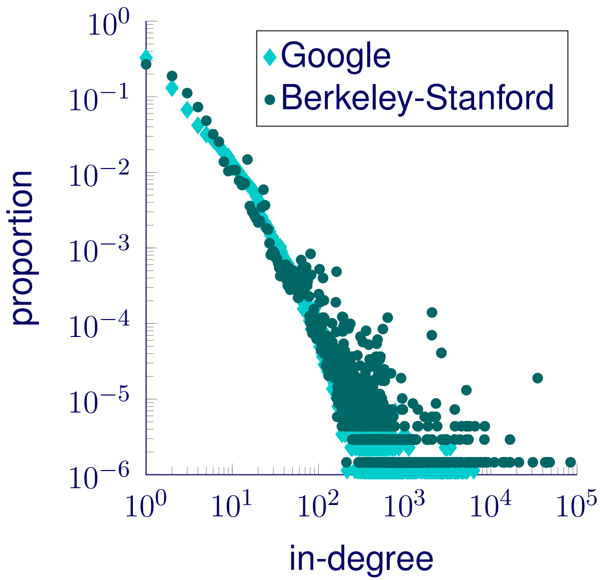


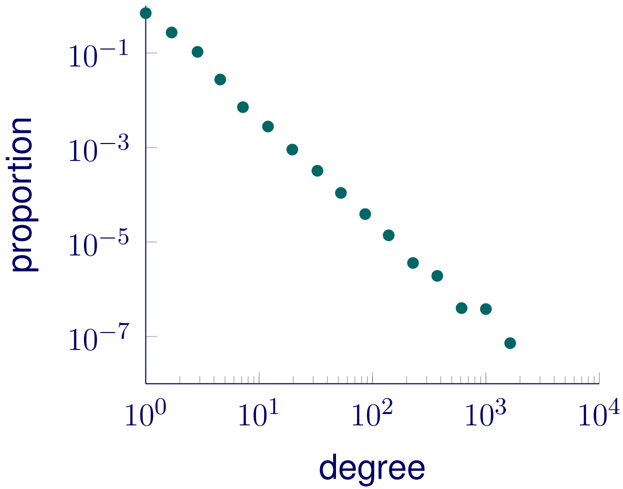


Yeast protein interaction network

Internet topology in 2004

Scale-free paradigm





Loglog plot degree sequences WWW in-degree and Internet

- ▷ Straight line: proportion p_k of vertices of degree k satisfies $p_k = ck^{-\tau}$.
- ▷ Empirical evidence: Often $\tau \in (2, 3)$ reported.

Competition

▷ Viral marketing aims to use social networks so as to excellerate adoption of novel products.

▷ Observation: Often one product takes almost complete market. Not always product of best quality:

Why?

▷ **Aim:** Explain this phenomenon, and relate it to network structure as well as spreading dynamics.

▷ Setting:
– Model social network as random graph;
– Model dynamics as competing rumors spreading through network, where vertices, once occupied by certain type, try to occupy their neighbors at (possibly) random and i.i.d. times:
▷ Fastest type might correspond to best product.

Competition and rumors

$\triangleright$ In absence competition, dynamics is rumor spread on graph.

$\triangleright$ Central role for spreading dynamics of such rumors=

shortest-weight routing on graph with i.i.d. random weights.

$\triangleright$ Main object of study: $\mathcal{C}_n$ is weight of smallest-weight path two uniform connected vertices:

$$\mathcal{C}_n = \min_{\pi : U_1 \to U_2} \sum_{e \in \pi} Y_e,$$

where π is path in G, while $(Y_e)_{e \in E(G)}$ are i.i.d. collection of exponential or deterministic weights.

Configuration model

Configuration model $\mathrm{CM}_n(\boldsymbol{d})$ is graph of fixed size in which degree sequence is prescribed:

$\triangleright$ n number of vertices;
$\triangleright$ $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ sequence of degrees is given.

$\triangleright$ This talk: take $(d_i)_{i \in [n]}$ independent and identically distributed (i.i.d.) random variables with power-law distribution.

Means that there exist $c_\tau > 0$ and $\tau \in (2,3)$ such that

$$\mathbb{P}(d_1 > k) = c_\tau k^{-\tau+1}(1 + o(1)).$$

Graph construction CM

$\triangleright$ Assign d_j half-edges to vertex j. Asume total degree

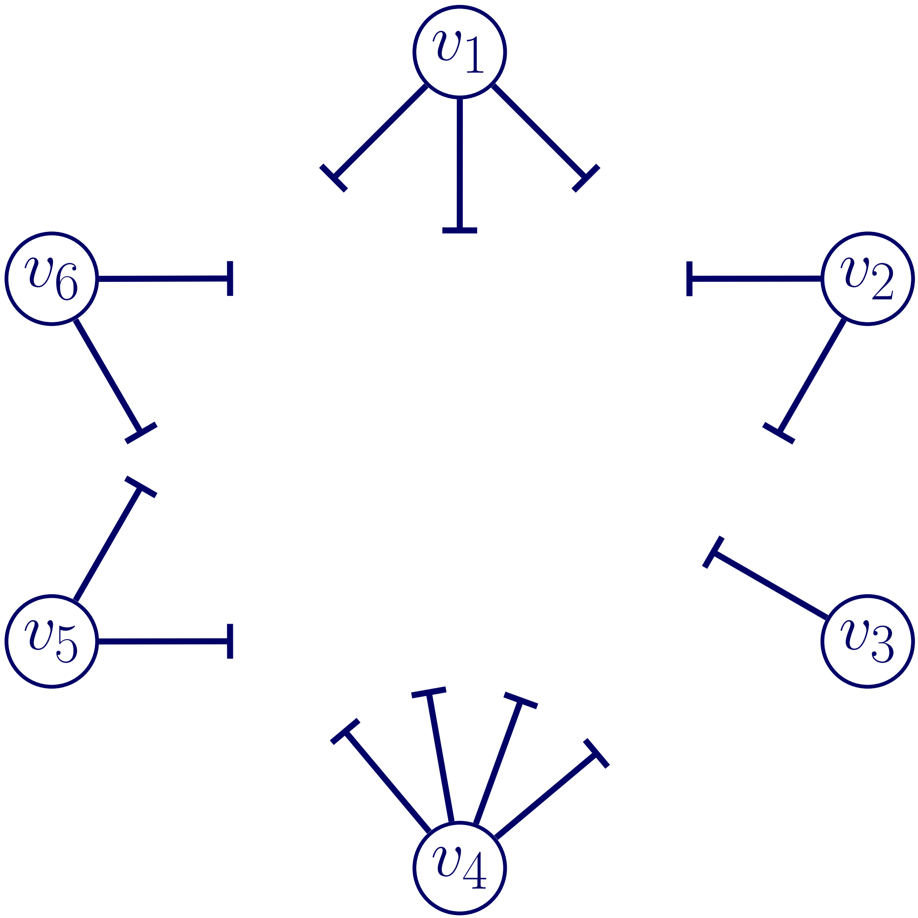
$$\ell_n = \sum_{i\in[n]} d_i$$

is even.

$\triangleright$ Pair half-edges to create edges as follows:
Number half-edges from 1 to ℓ_n in any order.
First connect first half-edge at random with one of other $\ell_n - 1$ half-edges.

$\triangleright$ Continue with second half-edge (when not connected to first) and so on, until all half-edges are connected.

$\triangleright$ Resulting graph is denoted by $\mathrm{CM}_n(\boldsymbol{d})$.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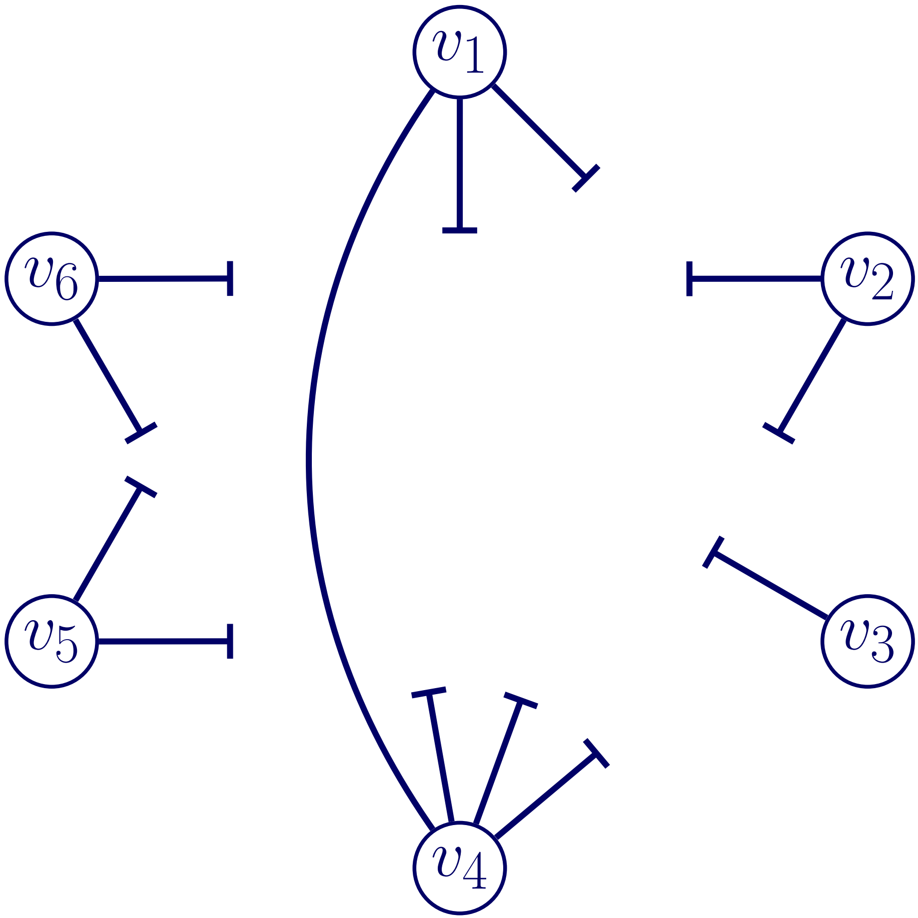

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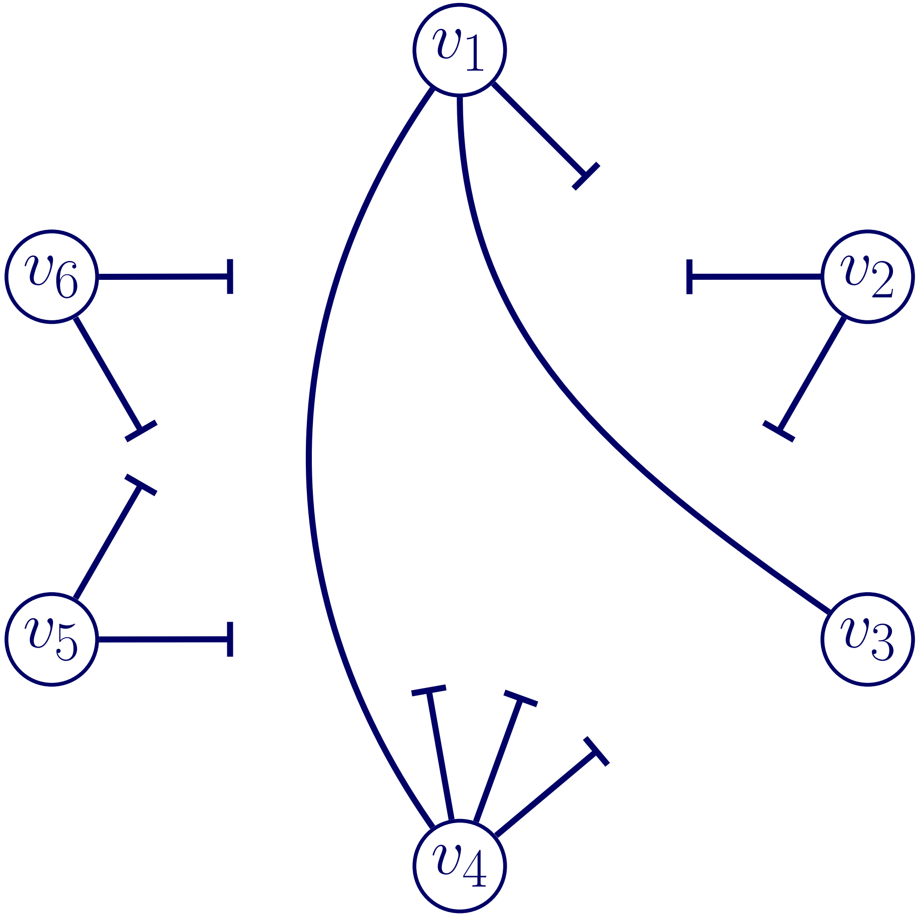

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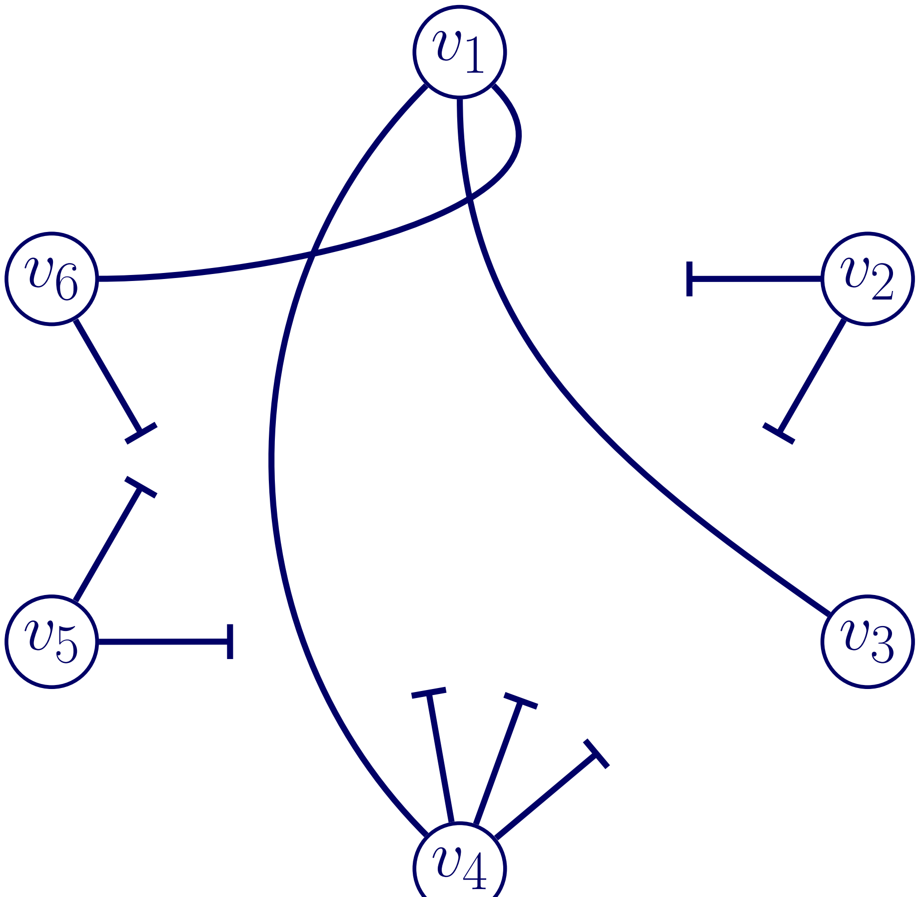
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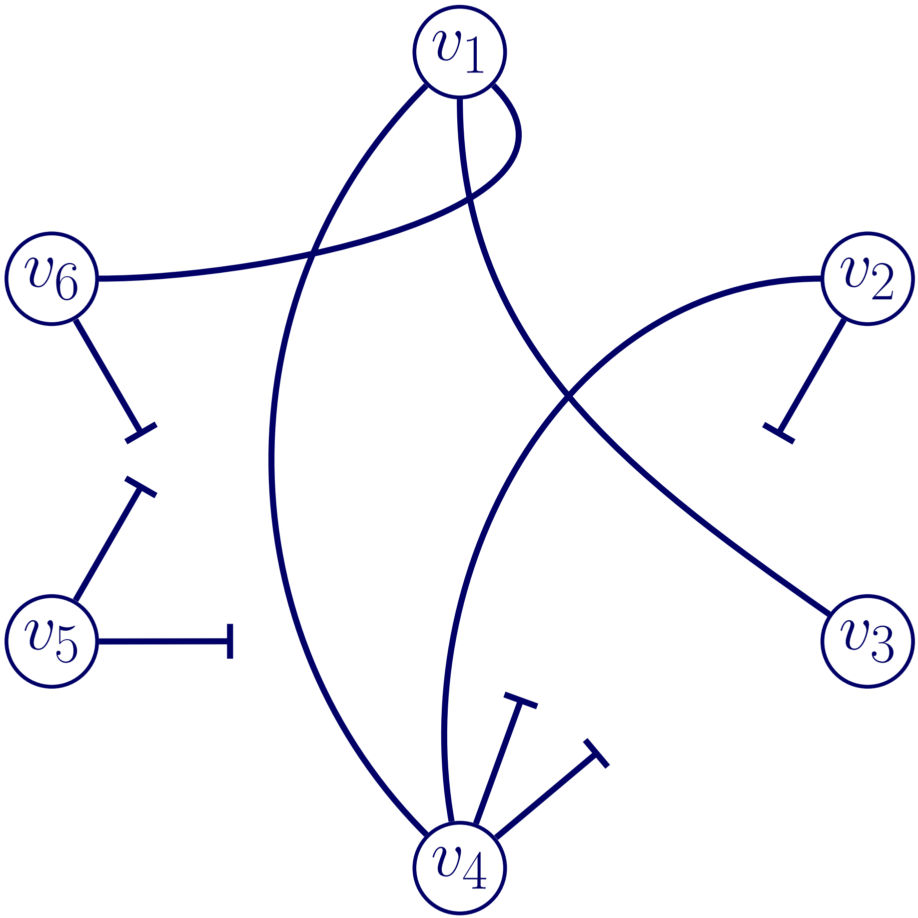

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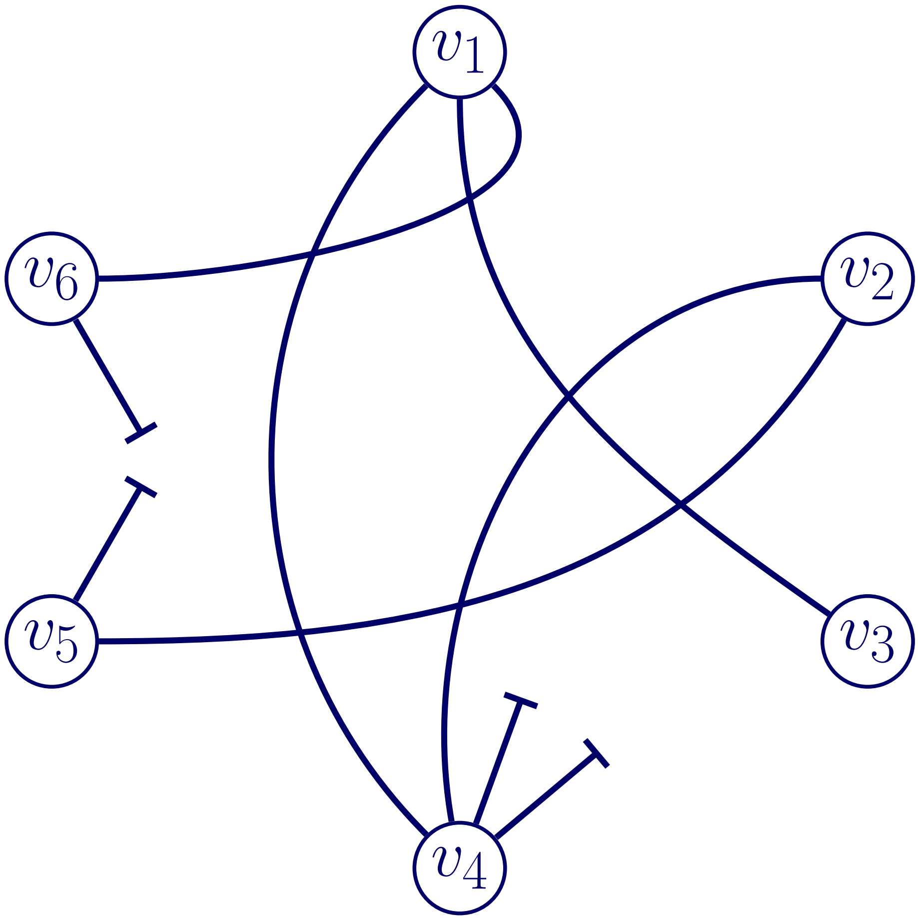

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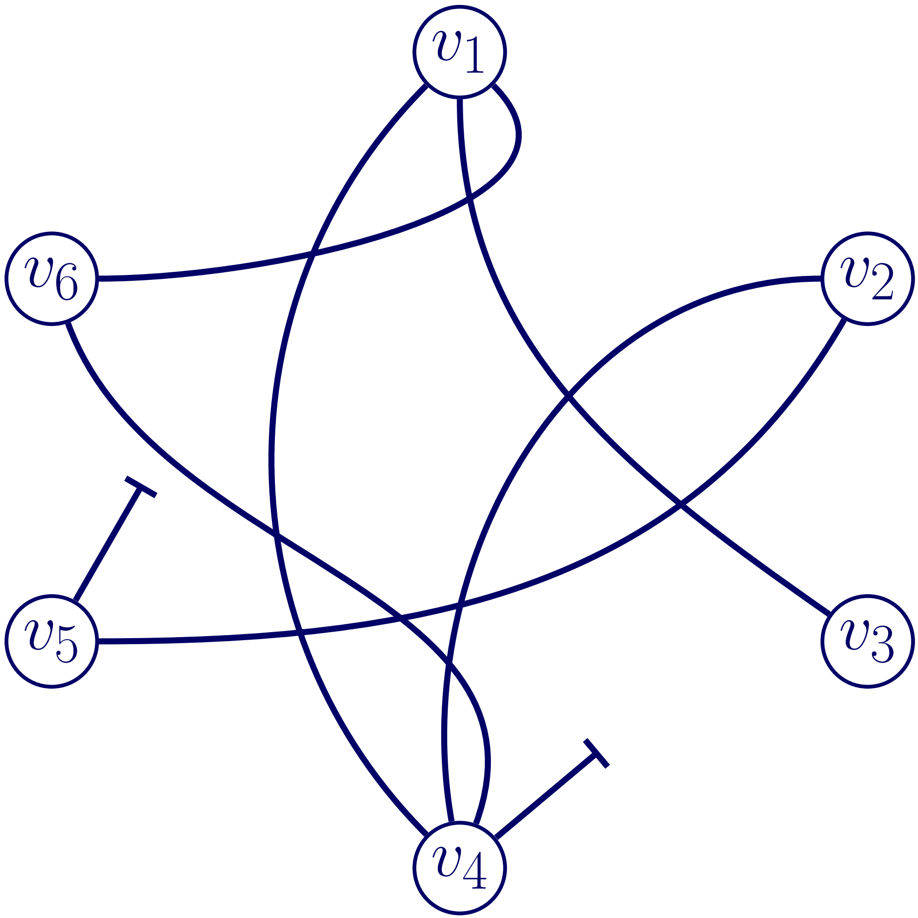
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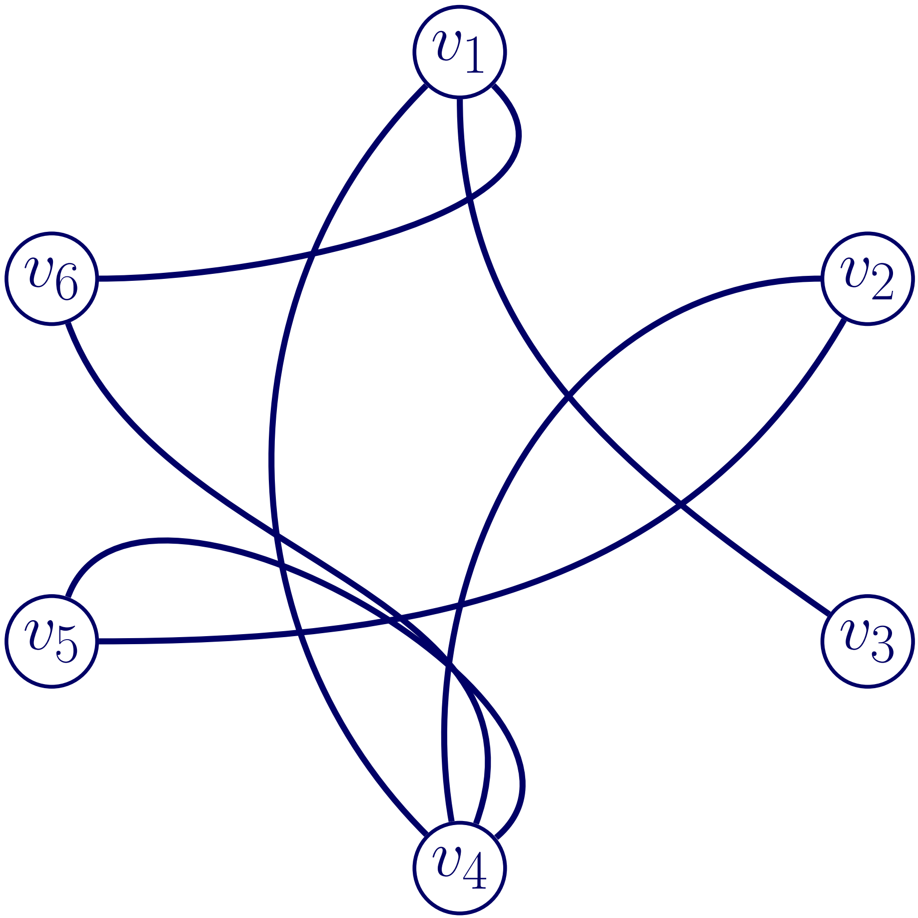
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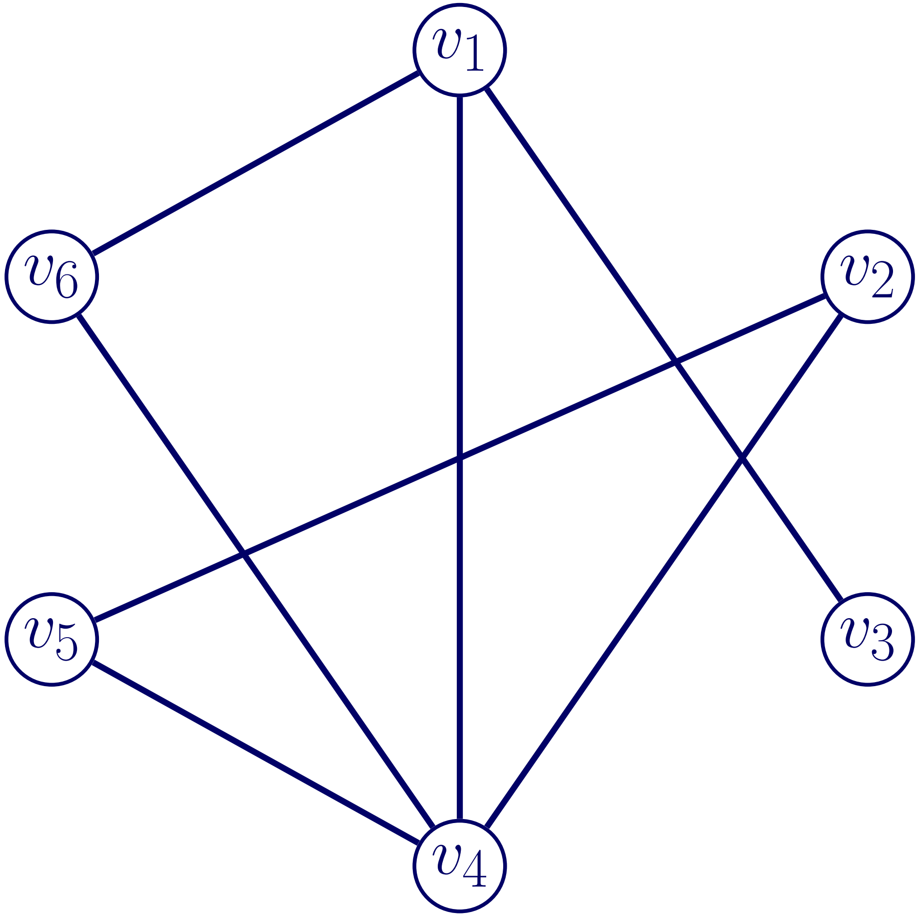

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Markovian spreading

Theorem 1. [Deijfen-vdH (2013)] Fix $\tau \in (2,3)$.
Consider competition model, where types compete for territory at fixed, but possibly unequal rates. Then, each of types wins majority vertices with positive probability:

$$\frac{N_1}{n} \xrightarrow{d} I \in \{0,1\}.$$

Number of vertices for losing type converges in distribution:

$$N_{\text{los}}(n) \xrightarrow{d} N_{\text{los}} \in \mathbb{N}.$$

The winner takes it all, the loser's standing small...

▷ Who wins is determined by location of starting point types:

Location, location, location!

Deterministic spreading

Theorem 2. [Baroni-vdH-Komjáthy (2014)] Fix $\tau \in (2,3)$.
Consider competition model, where types compete for territory with deterministic traversal times. Without loss of generality, assume that traversal time type 1 is 1, and of type 2 is $\lambda \geq 1$.

Fastest types wins majority vertices, i.e., for $\lambda > 1$,

$$\frac{N_1(n)}{n} \xrightarrow{\mathbb{P}} 1.$$

Number of vertices for losing type 2 satisfies that there exists random variable Z s.t.

$$\frac{\log(N_2(n))}{(\log n)^{2/(\lambda+1)} C_n} \xrightarrow{d} Z.$$

▷ Here, C_n is some random oscillatory sequence.

Deterministic spreading

Theorem 3. [vdH-Komjáthy (2014)] Fix $\tau \in (2,3)$.
Consider competition model, where types compete for territory with deterministic equal traversal times.

▷ When starting locations of types are sufficiently different,

$$\frac{N_1(n)}{n} \xrightarrow{d} I \in \{0,1\},$$

and number of vertices for losing type satisfies

$$\frac{\log(N_{\mathrm{los}}(n))}{C_n \log n} \xrightarrow{\mathbb{P}} 1,$$

where $C_n \in (0,1)$ whp.
▷ When starting locations are sufficiently similar, **coexistence** occurs, i.e., there exist $0 < c_1, c_2 < 1$ s.t. whp

$$\frac{N_1(n)}{n}, \frac{N_2(n)}{n} \in (c_1, c_2).$$

Neighborhoods CM

$\triangleright$ Important ingredient in proof is description local neighborhood of uniform vertex $U_1 \in [n]$. Its degree has distribution $D_{U_1} \stackrel{d}{=} D$.

$\triangleright$ Take any of D_{U_1} neighbors a of U_1. Law of number of forward neighbors of a, i.e., $B_a = D_a - 1$, is approximately

$$\mathbb{P}(B_a = k) \approx \frac{(k+1)}{\sum_{i\in[n]} d_i} \sum_{i\in[n]} \mathbb{1}_{\{d_i=k+1\}} \xrightarrow{\mathbb{P}} \frac{(k+1)}{\mathbb{E}[D]}\mathbb{P}(D = k+1).$$

Equals size-biased version of D minus 1. Denote this by $D^\star - 1$.

Local tree-structure CM

$\triangleright$ Forward neighbors of neighbors of U_1 are close to i.i.d. Also forward neighbors of forward neighbors have asymptotically same distribution...

$\triangleright$ Conclusion: Neighborhood looks like branching process with offspring distribution $D^\star - 1$ (except for root, which has offspring $D.$)

$\triangleright$ $\tau \in (2,3):$ Infinite-mean BP, which has double exponential growth of generation sizes:

$$(\tau-2)^k \log(Z_k \vee 1) \xrightarrow{a.s.} Y \in (0,\infty).$$

Graph distances CM: Theorems 2+3

H_n is graph distance between uniform pair of vertices in $\mathrm{CM}_n(\boldsymbol{d})$.

Theorem 4. [vdHHZ07, Norros-Reittu 04]. Fix $\tau \in (2,3)$. Then,

$$\frac{H_n}{\log\log n} \xrightarrow{\mathbb{P}} \frac{2}{|\log(\tau-2)|},$$

and fluctuations are tight, but do not converge in distribution.

$\triangleright$ In absence of competition, it takes each of types about $\frac{\log\log n}{|\log(\tau-2)|}$ steps to reach vertex of maximal degree.

$\triangleright$ Type that reaches vertices of highest degrees (=hubs) first wins. When $\lambda > 1$, fastest type wins whp.

$\triangleright$ Coexistence occurs when both vertices find hubs at same time.

Proof Theorem 1

Theorem 5. [Bhamidi-vdH-Hooghiemstra AoAP10]. Fix $\tau \in (2,3)$. Then,

$$\mathcal{C}_n \xrightarrow{d} \mathcal{C}_\infty,$$

for some limiting random variable $\mathcal{C}_\infty$:

Super efficient rumor spreading.

$\triangleright$ $\mathcal{C}_\infty \stackrel{d}{=} V_1 + V_2,$ where V_1, V_2 are i.i.d. explosion times of CTBP starting from vertices U_1, U_2. Then,

$$I = \mathbb{1}_{\{V_1 < \lambda V_2\}}.$$

Law of N_{los} much more involved, as competition changes dynamics after winning type has found hubs.

Conclusions

▷ Random graph models:
Used to explain properties of real-world networks:

Universality?

▷ Competition: Who wins depends sensitively on competition dynamics, as well as on topology network. When hubs dominate, types that gets there first whp occupies majority graph.

Other laws traversal times?

Believe that winner take it all phenomenon is universal for explosive setting for infinite-variance degrees.

▷ Book on random graphs (to appear 2016):
Random Graphs and Complex Networks
`http://www.win.tue.nl/~rhofstad/NotesRGCN.html`

Literature

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