Asymptotic analysis of the distribution of the sojourn time for a random walk above a receding boundary

Borisov I.S., Shefer E.I.

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Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables with mean zero and finite variance $\sigma^2 := \mathbf{E}\xi_1^2 > 0$. Put $S_0 := 0$. Denote $S_k := \sum_{i=1}^{k} \xi_i$, $k = 1, 2, ..., n$. We define the sojourn time for a trajectory of the random walk $\{S_1, \ldots, S_n\}$ above a level $xg(\cdot)$ as the random variable

$$\tau_n(xg) := \sum_{k=1}^{n} I\{S_k \geq xg(k/n)\},$$

where $I(\cdot)$ is the indicator of an event, $x \equiv x(n) \to \infty$ as $n \to \infty$ characterizes the speed of the boundary moving, and a bounded positive function $g(t)$, $t \in (0, 1]$, determines the configuration of the curvilinear boundary in dependence on time.

The mean sojourn time asymptotics

First, we study the asymptotic behavior of the mean

$$\mathbf{E}\tau_n(xg) = \sum_{k=1}^{n} \mathbf{P}\{S_k > xg(k/n)\}$$

as $n \to \infty$.
Introduce the function

$$G(t) := g^2(t)/t, \quad t \in (0,1].$$

We put $G(t) := g^2(1)/t$ for all t on a right semi-neighborhood of the point $t = 1$. Suppose that there exists a point $t_0 \in (0,1]$ in which the function $G(\cdot)$ attains its minimum and the following four conditions hold:

The mean sojourn time asymptotics

1) For some $0 < r \le g(t_0)$,

$$G(t) \ge \max\left\{c_0(t), \frac{r^2}{t}\right\} \text{ for all } \ t \in (0,1],$$

where a positive function $c_0(t)$ decreases on $(0, t_0)$ and increases on $(t_0, 1]$, and $c_0(t_0) = G(t_0)$;
2) The function $G(t)$ is m_1-times continuously differentiable on a left semi-neighborhood of the point t_0, and is m_2-times differentiable on a right semi-neighborhood of t_0; here m_1 and m_2 are the respective orders of the left and right first nonzero derivatives of $G(t)$ at the point $t = t_0$;
3) $x/\sqrt{n} \to \infty$ and $x/n^{1-\gamma_m} \to 0$ as $n \to \infty$, where

$$\gamma_m := \frac{1 - 1/m}{3 - 2/m}, \quad m := \max\{m_1, m_2\};$$

4) $\mathbf{E}e^{\lambda|\xi_1|} < \infty$ for some $\lambda > 0$.

The mean sojourn time asymptotics

Theorem 1. Under the conditions 1—4, and $n \to \infty$,

$$\mathbf{E}\tau_n(xg) \sim M(G, t_0, m_1, m_2) \frac{n(2m!)^{1/m}\sqrt{t_0}}{\sqrt{2\pi} g(t_0)}$$

$$\times \left(\frac{\sigma}{\beta_n}\right)^{1+2/m} \Gamma\left(\frac{m+1}{m}\right) \exp\left\{-nt_0\Lambda\left(\frac{\beta_n g(t_0)}{\sqrt{n} t_0}\right)\right\}, \qquad (1)$$

where

$$\beta_n := \frac{x}{\sqrt{n}}, \quad \Gamma(z) := \int_0^\infty y^{z-1} e^{-y}\, dy, \quad z > 0,$$

$\Lambda(z)$ is the deviation function (Cramér's function) of the random variable ξ_1, i.e.,

$$\Lambda(z) := \sup_t \left\{tz - \log \psi_{\xi_1}(t)\right\} \geq 0, \quad \text{where} \quad \psi_{\xi_1}(t) := \mathbf{E}e^{t\xi_1}$$

(here $\log(\infty) = \infty$ and $c - \infty = -\infty$ by definition),

The mean sojourn time asymptotics

$$M(G, t_0, m_1, m_2) :=$$

$$\begin{cases} \dfrac{1}{\left|G^{(m_1)}(t_0-0)\right|^{1/m_1}} & \text{if } \; t_0 \in (0,1) \text{ and } m_1 > m_2, \\ \dfrac{1}{\left|G^{(m_2)}(t_0+0)\right|^{1/m_2}} & \text{if } \; t_0 \in (0,1) \text{ and } m_1 < m_2, \\ \dfrac{1}{\left|G^{(m_1)}(t_0-0)\right|^{1/m_1}} + \dfrac{1}{\left|G^{(m_2)}(t_0+0)\right|^{1/m_2}} & \text{if } \; t_0 \in (0,1) \text{ and } m_1 = m_2, \\ \dfrac{1}{\left|G^{(m_1)}(1-0)\right|^{1/m_1}} & \text{if } \; t_0 = 1. \end{cases}$$

If $m \geq 2$ then the argument of the exponential function in (1) can be replaced with the following expression:

$$-\frac{\beta_n^2 g^2(t_0)}{2t_0\sigma^2} + \frac{\mathbf{E}\xi_1^3 \beta_n^3 g^3(t_0)}{6t_0^2\sigma^6\sqrt{n}}.$$

The particular case $g(\cdot) \equiv 1$ of Theorem 1, which is included in the case $t_0 = 1$ and $m = 1$, proved by Lotov and Tarasenko (2015).
Theorem 1 considered the case where the graph of the function $g(t)$ touched the rotated parabola at one point (or at a finite number of points) of the semi-open interval $(0, 1]$. The following theorem considers the case in which, on a nondegenerate subinterval within $(0, 1]$, the function $g(t)$ coincides with the rotated parabola or, in other words, the function $G(t)$ is identically equal to its minimum value on some nondegenerate interval $[t_1, t_2] \subset (0, 1]$.
We assume that the following conditions similar to conditions 1 and 2 of Theorem 1, are satisfied:

1′) There exists a positive $r \le g(t_1)$ such that

$$G(t) \ge \max\left\{c_0(t), \frac{r^2}{t}\right\} \text{ for all } \; t \in (0,1],$$

where a positive function $c_0(t)$ decreases on $[0, t_1)$ and increases on $(t_2, 1]$, and $c_0(t) = G(t_1)$ for all $t \in [t_1, t_2]$;
2′) The function $G(t)$ is m_1-times continuously differentiable in a left semi-neighborhood of the point t_1 and is m_2-times continuously differentiable in a right semi-neighborhood of the point t_2; here m_1 and m_2 are the orders of the first nonzero left and right derivatives of the function $G(t)$ at the points t_1 and t_2, respectively.

To formulate Theorem 2 we introduce the following notation:

$$\Lambda_0(z) := \Lambda(z) - z^2/2\sigma^2.$$

Notice that $z^2/2\sigma^2$ is the first term of the Cramér series, i.e., the first nonzero term of the Taylor expansion of the deviation function $\Lambda(z)$ in a neighborhood of zero.

The mean sojourn time asymptotics

Theorem 2. Under the conditions $1'$, $2'$, 3, and 4,

$$\mathbf{E}\tau_n(xg) \sim \frac{n^{3/2}\sigma\sqrt{t_1}}{\sqrt{2\pi}xg(t_1)} \exp\left\{-\frac{x^2g^2(t_1)}{2nt_1\sigma^2}\right\} J_n(t_1, t_2),$$

as $n \to \infty$, where

$$J_n(t_1, t_2) := \int_{t_1}^{t_2} \exp\left\{-nt\Lambda_0\left(\frac{xg(t_1)}{n\sqrt{t_1 t}}\right)\right\} dt.$$

In case $m \geq 2$,

$$J_n(t_1, t_2) \sim \int_{t_1}^{t_2} \exp\left\{\frac{g^3(t_1)x^3\mathbf{E}\xi_1^3}{6\sigma^6 t_1^{3/2} n^2\sqrt{t}}\right\} dt,$$

in particular,

$$J_n(t_1, t_2) \sim \begin{cases} t_2 - t_1, & \text{if } \mathbf{E}\xi_1^3 = 0 \text{ or } \frac{x^3}{n^2} \to 0, \\ \frac{12n^2t_1^3\sigma^6}{x^3g^3(t_1)\mathbf{E}\xi_1^3} \exp\left\{\frac{x^3g^3(t_1)\mathbf{E}\xi_1^3}{6n^2t_1^2\sigma^6}\right\}, & \text{if } \mathbf{E}\xi_1^3 > 0 \text{ and } \frac{x^3}{n^2} \to \infty, \\ \frac{12n^2t_2^3\sigma^6}{x^3g^3(t_2)|\mathbf{E}\xi_1^3|} \exp\left\{\frac{x^3g^3(t_2)\mathbf{E}\xi_1^3}{6n^2t_2^2\sigma^6}\right\}, & \text{if } \mathbf{E}\xi_1^3 < 0 \text{ and } \frac{x^3}{n^2} \to \infty. \end{cases}$$

The tail distribution asymptotics

Next we assume that $g(t) \equiv 1$, i.e. we consider the case when the moving boundary is straight-line. Our goal is to study the asymptotic behavior, as $n \to \infty$, of the distribution tail of the normed random variable $\tau_n(x)/n$, i.e, of the probability

$$\mathbf{P}\{\tau_n(x)/n \geq y\} = \mathbf{P}\{\tau_n(x) \geq ny\}$$

for any fixed $y \in (0,1)$ in the case where $x \equiv x(n)$ tends to infinity in the moderate large deviations range.

The tail distribution asymptotics

We assume the following moment restrictions on ξ_1 as well as on the moving speed of x: For some $\lambda > 0$ and $r \in [1, 2]$, it is fulfilled:
1) $\mathbf{E}e^{\lambda|\xi_1|} < \infty$ and $\mathbf{E}\{e^{\lambda\xi_1^r} I(\xi_1 \geq 0)\} < \infty$;
2) $\frac{x}{\sqrt{n}} \to \infty$, $x = o\big(\min\{n^{(r+1)/(r+2)}, (n/\log n)^{3/4}\}\big)$.
It is easy to see that condition 1 for $r = 1$ coincides with Cramér's condition and, in this case, the range of deviations has the order $x = o(n^{2/3})$.

The tail distribution asymptotics

Theorem 3. *Under conditions* 1 *and* 2, *for any fixed* $y \in (0,1)$ *and* $n \to \infty$, *the following asymptotic relation is valid*:

$$\mathbf{P}\{\tau_n(x) \geq ny\} \sim \frac{2(1-y)^{3/2}}{\pi\sqrt{y}} \frac{n\sigma^2}{x^2} \exp\left\{-n(1-y)\Lambda\left(\frac{x}{n(1-y)}\right)\right\}; \tag{2}$$

in case $r = 1$, *the argument of the exponential function in* (1) *can be replaced with* $-x^2/(2\sigma^2 n(1-y))$, *and in case* $r \in (1,2]$, *with*

$$-\frac{x^2}{2\sigma^2 n(1-y)} + \frac{x^3 \mathbf{E}\xi_1^3}{6\sigma^6 n^2 (1-y)^2}.$$

Geometrical and Exponential cases

We consider two special cases of distribution of ξ_1:
(I): The *one-sided exponential* distribution

$$\mathbf{P}\{\xi_1 \in dt\} = p\alpha e^{-\alpha t},\ t > 0,$$

where $p > 0, \alpha > 0$. Here another component of the distribution (on the non-positive real line) can be arbitrary at that satisfying Cramér's condition.
(II): The *one-sided generalized geometric* distribution

$$\mathbf{P}\{\xi_1 = k\} = \alpha p^{k-1},\ k \in \mathbb{N},$$

where $p \geq 0, \alpha > 0$ under the agreement $0^0 = 1$. Here another component of the distribution must be arithmetic satisfying Cramér's condition. Notice that, in case $p = 0$, we deal with an upper semicontinuous random walk.
The condition $\mathbf{E}\xi_1 = 0$ still has to be fulfilled.

Notice that, for the above-mentioned distributions, condition 1 is not fulfilled for $r > 1$. In this case, the phenomenon of "memorylessness" is well known, in which the distribution of overshooting an arbitrary level of a random walk coincides with the distribution of ξ_1. This remarkable property allows us to improve the result of Theorem 3 in this special case replacing condition 2 with the following:
2') $\frac{x}{\sqrt{n}} \to \infty$, $x = o\big((n/\log n)^{3/4}\big)$.
Theorem 4. *Let ξ_1 have a distribution from Classes* $\mathrm{(I)}$ *or* $\mathrm{(II)}$*. Then under conditions* 1 *and* $2'$*, for any fixed $y \in (0,1)$ and $n \to \infty$, the following asymptotic relation is valid*:

$$\mathbf{P}\{\tau_n(x) \geq ny\} \sim \frac{2(1-y)^{3/2}}{\pi\sqrt{y}} \frac{n\sigma^2}{x^2} \exp\left\{-n(1-y)\Lambda\left(\frac{x}{n(1-y)}\right)\right\}.$$

Simple random walk

Now we consider another special case called a *simple symmetric random walk*, i.e., if $\xi_1 = \pm 1$ with probabilities $1/2$. We proved that, in this special case, the asymptotic relation (2) is valid in the whole moderate large deviations range $xn^{-1/2} \to \infty$ and $x = o(n)$.

Theorem 5. *For a simple symmetric random walk and all sequences $x \equiv x(n)$ satisfying the conditions $\frac{x}{\sqrt{n}} \to \infty$ and $\frac{x}{n} \to 0$ as $n \to \infty$, the following asymptotic relation is valid for any fixed $y \in (0,1)$:*

$$\mathbf{P}\{\tau_n(x) \geq ny\} \sim \frac{2(1-y)^{3/2}}{\pi\sqrt{y}} \frac{n}{x^2} \exp\left\{-n(1-y)\Lambda\left(\frac{x}{n(1-y)}\right)\right\},$$

where

$$\Lambda(z) = \frac{1+z}{2}\log(1+z) + \frac{1-z}{2}\log(1-z), \quad |z| < 1.$$

References

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